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The Geometrical Aberrations of General Electron Optical Systems II. The Primary (Third Order) Aberrations of Orthogonal Systems, and the Secondary (Fifth Order) Aberrations of Round Systems

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THE GEOMETRICAL ABERRATIONS OF GENERAL ELECTRON OPTICAL SYSTEMS‡

II. THE PRIMARY (THIRD ORDER) ABERRATIONS OF ORTHOGONAL SYSTEMS, AND THE SECONDARY (FIFTH ORDER) ABERRATIONS OF ROUND SYSTEMS

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CONTENTS

The theory of characteristic functions, developed by Sturrock for electron optics, is used to calculate the primary aberrations of rectilinear orthogonal systems of the most general kind. In the second part, the secondary aberrations of round systems are calculated with the aid of Sturrock's second-order perturbation characteristic functions. A proof of the equivalence of the aberration formulae obtained by Melkich, using the variation of parameters method, and those obtained below is offered in an appendix.

Historical introduction

The primary aberrations of rectilinear orthogonal systems are discussed in detail in only one publication, a Berlin Dissertation by Alexander Melkich (1944, published 1947). Melkich uses the method of variation of parameters to calculate the form of the aberration coefficients in a (twist-free§) orthogonal system which need not be stigmatic in first order. The form of the primary aberration coefficients is somewhat complex, and in an appendix, I demonstrate that his expressions are equivalent to those obtained with the aid of characteristic functions.

Apart from passing allusions and brief discussions of the effects to be expected from secondary aberrations if primary aberrations could be corrected, the fifth-order aberrations of any kind of electron optical system have been discussed only by U My-Chzhen' (1957). The latter analyses the fifth-order aberrations of rotationally symmetrical systems, and sets out the expressions for spherical aberration and distortion in full. (The development of the theory of fifth-order aberrations in 'light optics' is traced in Hawkes (1963, pp. 128–9) from Schwarzschild's early work up to Focke's study which appeared in 1951.)

‡ The present work is substantially equivalent to the third chapter of a Cambridge Doctoral Dissertation, entitled 'The aberrations of electron optical systems in the absence of rotational symmetry' (1963).

§ See §3·2 of part I of the present work (Hawkes 1965 *a*).

A. The primary aberrations of rectilinear‡ orthogonal systems

The general case

The refractive index of an electron optical medium characterized by an electric field, scalar potential $\phi(X, Y, z)$ and a magnetic field, vector potential $\mathbf{A}(X, Y, z)$ is

$$n = \sqrt{\{\phi(1+\epsilon\phi)\}} - \sqrt{(e/2m_0)}\, \mathbf{A}\,.\,\mathbf{s},$$

in which $\mathbf{s}$ is a unit vector along the ray in question, $-e$ is the charge on the electron, m_0 its rest mass, and $\epsilon = e/2m_0c^2$ is a relativistic correction constant (c denotes the velocity of light). If we select the axis of the system as the z axis of a system of Cartesian co-ordinates, the quantity $n\,\mathrm{d}s$ which occurs in the expression of Fermat's principle can be replaced by $m\,\mathrm{d}z$, where

$$m = \sqrt{\{\phi(1+\epsilon\phi)\,(1+X'^2+Y'^2)\}} - \sqrt{(e/2m_0)}\,(A_X X' + A_Y Y' + A_z);$$

X' and Y' denote $\mathrm{d}X/\mathrm{d}z$ and $\mathrm{d}Y/\mathrm{d}z$ respectively.

The electrostatic and magnetic potentials can be expanded in terms of the off-axial co-ordinates X, Y, and functions of z only, as follows:§

$$\left.\begin{aligned}\phi(X, Y, z) = {} & \Phi(z) - \tfrac{1}{4}[\Phi''(z) - D(z)]\,X^2 + P(z)\,XY - \tfrac{1}{4}[\Phi''(z) + D(z)]\,Y^2,\\ & + [\tfrac{1}{64}\Phi^{(\mathrm{iv})}(z) - \tfrac{1}{48}D''(z) + D_1(z)]\,X^4 - [\tfrac{1}{12}P''(z) - 4P_1(z)]\,X^3Y\\ & + [\tfrac{1}{32}\Phi^{(\mathrm{iv})}(z) - 6D_1(z)]\,X^2Y^2 - [\tfrac{1}{12}P''(z) + 4P_1(z)]\,XY^3\\ & + [\tfrac{1}{64}\Phi^{(\mathrm{iv})}(z) + \tfrac{1}{48}D''(z) + D_1(z)]\,Y^4,\end{aligned}\right\} \quad (\mathrm{A}\,1\,a)$$

$$\left.\begin{aligned}A_X(X, Y, z) = {} & -\Omega(z)\,Y + \tfrac{1}{4}[\Omega''(z) + \Delta'(z)]X^2Y + \tfrac{1}{2}Q'(z)\,XY^2 + \tfrac{1}{12}[\Omega''(z) - \Delta'(z)]\,Y^3,\\ A_Y = {} & 0,\\ A_z = {} & \tfrac{1}{2}Q(z)\,X^2 - \tfrac{1}{2}[\Omega'(z) + \Delta(z)]\,XY - \tfrac{1}{2}Q(z)\,Y^2 - [\tfrac{1}{48}Q''(z) - Q_1(z)]\,X^4\\ & + \tfrac{1}{16}[\Omega'''(z) + \tfrac{1}{12}\Delta''(z) - 4\Delta_1(z)]\,X^3Y + [\tfrac{1}{8}Q''(z) - 6Q_1(z)]\,X^2Y^2\\ & + [\tfrac{1}{48}\Omega'''(z) + 4\Delta_1(z)]\,XY^3 + [\tfrac{1}{48}Q''(z) + Q_1(z)]\,Y^4.\end{aligned}\right\} \quad (\mathrm{A}\,1\,b)$$

We denote terms of m in z alone by $m^{(0)}$, and terms of second and fourth degree in X and Y and their derivatives by $m^{(2)}$ and $m^{(4)}$ respectively; writing η for $\sqrt{(e/2m_0)}$, we find

$$m^{(0)} = \sqrt{\{\Phi(1+\epsilon\Phi)\}}, \quad (\mathrm{A}\,2\,a)$$

$$\begin{aligned} m^{(2)} = {} & -\frac{1}{8}\frac{1+2\epsilon\Phi}{\sqrt{\{\Phi(1+\epsilon\Phi)\}}}\Phi''(X^2+Y^2)\\ & + \left[\frac{1}{8}\frac{1+2\epsilon\Phi}{\sqrt{\{\Phi(1+\epsilon\Phi)\}}}D - \tfrac{1}{2}\eta Q\right](X^2-Y^2)\\ & + \left[\frac{1}{2}\frac{1+2\epsilon\Phi}{\sqrt{\{\Phi(1+\epsilon\Phi)\}}}P + \tfrac{1}{2}\eta\Delta\right]XY\\ & + \tfrac{1}{2}\sqrt{\{\Phi(1+\epsilon\Phi)\}}\,(X'^2+Y'^2) - \tfrac{1}{2}\eta\Omega(XY' - X'Y), \end{aligned} \quad (\mathrm{A}\,2\,b)$$

‡ By a 'rectilinear' system, we understand a system with a straight optic axis.

§ The notation is very similar to that employed by Glaser (1956), except that $(-\partial\phi_m/\partial z)_{x=y=0}$ is denoted by $\Omega(z)$.

$$\begin{aligned} m^{(4)} = {} & F(X^2+Y^2)^2+G(X^2-Y^2)^2 \\ & +HXY(X^2+Y^2)+IXY(X^2-Y^2) \\ & +J(X^4-Y^4)+K(X'^2+Y'^2)(X^2+Y^2) \\ & +L(X'^2+Y'^2)(X^2-Y^2)+M(X'^2+Y'^2)XY \\ & +N(X'^2+Y'^2)^2+RXY(XY'-X'Y) \\ & +SXY\frac{\mathrm{d}}{\mathrm{d}z}(X^2-Y^2)+TXY\frac{\mathrm{d}}{\mathrm{d}z}(X^2+Y^2) \end{aligned} \tag{A 2c}$$

in which

$$\left.\begin{aligned} F = {} & \frac{1}{128}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}\Phi^{(\mathrm{iv})}-\frac{1}{128}\frac{(\Phi'')^2}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}}-\frac{1+2\epsilon\Phi}{2\surd\{\Phi(1+\epsilon\Phi)\}}D_1 \\ & -\frac{1}{32}\frac{P^2}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}}+\eta Q_1, \\ G = {} & \frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}D_1-\frac{1}{128}\frac{D^2}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}}+\frac{1}{32}\frac{P^2}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}}-2\eta Q_1, \\ H = {} & -\frac{1}{24}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}P''+\frac{1}{16}\frac{P\Phi''}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}}-\frac{1}{12}\eta\Delta'', \\ I = {} & \frac{2(1+2\epsilon\Phi)}{\surd\{\Phi(1+\epsilon\Phi)\}}P_1-\frac{1}{16}\frac{DP}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}}-\frac{1}{16}\eta\Omega'''+4\eta\Delta_1, \\ J = {} & -\frac{1}{96}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}D''+\frac{1}{64}\frac{D\Phi''}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}}+\frac{1}{48}\eta Q'', \\ K = {} & -\frac{1}{16}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}\Phi'', \quad L=\frac{1}{16}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}D, \\ M = {} & \frac{1}{4}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}P, \quad N=-\tfrac{1}{8}\surd\{\Phi(1+\epsilon\Phi)\}, \\ R = {} & \tfrac{1}{4}\eta Q', \quad S=-\tfrac{1}{8}\eta\Omega'', \quad T=-\tfrac{1}{8}\eta\Delta'. \end{aligned}\right\} \tag{A 3}$$

The term in $(XY'-XY')$ which appears in $m^{(2)}$ can be eliminated by rotating the axes through an angle Θ, such that

$$\frac{\mathrm{d}\Theta}{\mathrm{d}z}=\frac{\eta}{2}\frac{\Omega}{\surd\{\Phi(1+\epsilon\Phi)\}} \tag{A 4}$$

and in the new co-ordinates, (x, y), we find

$$m^{(0)}=\surd\{\Phi(1+\epsilon\Phi)\}, \tag{A 5a}$$

$$\begin{aligned} m^{(2)} = {} & \left[-\frac{1}{8}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}\Phi''-\frac{\eta^2}{8}\frac{\Omega^2}{\surd\{\Phi(1+\epsilon\Phi)\}}\right](x^2+y^2) \\ & +\left[\left(\frac{1}{8}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}D-\tfrac{1}{2}\eta Q\right)\cos 2\Theta\right. \\ & \left.+\left(\frac{1}{4}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}P+\tfrac{1}{4}\eta\Delta\right)\sin 2\Theta\right](x^2+y^2) \\ & +\left[-\left(\frac{1}{4}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}D-\eta Q\right)\sin 2\Theta\right. \\ & \left.+\left(\frac{1}{2}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}P+\tfrac{1}{2}\eta\Delta\right)\cos 2\Theta\right]xy \\ & +\tfrac{1}{2}\surd\{\Phi(1+\epsilon\Phi)\}(x'^2+y'^2), \end{aligned} \tag{A 5b}$$

$$
\begin{aligned}
m^{(4)} = {} & [\hat{j}-\tfrac{1}{2}\hat{k}\sin(4\Theta+\Pi)]\,(x^2+y^2)^2+\hat{k}\sin(4\Theta+\Pi)\,(x^2-y^2)^2 \\
& +2\hat{k}\cos(4\Theta+\Pi)\,xy(x^2-y^2)+2\hat{m}\cos(2\Theta+\Sigma)\,xy(x^2+y^2) \\
& +\hat{m}\sin(2\Theta+\Sigma)\,(x^4-y^4)+\hat{n}(x^2+y^2)\,(x'^2+y'^2) \\
& +2\hat{n}\Theta'(x^2+y^2)\,(xy'-x'y)+\hat{q}\sin(2\Theta+\Xi)\,(x^2-y^2)\,(x'^2+y'^2) \\
& +2\hat{q}\cos(2\Theta+\Xi)\,xy(x'^2+y'^2)+\hat{r}\sin(2\Theta+\Lambda)\,(x^2-y^2)\,(xy'-x'y) \\
& +2\hat{r}\cos(2\Theta+\Lambda)\,xy(xy'-x'y)+N(x'^2+y'^2)^2 \\
& +4N\Theta'(x'^2+y'^2)\,(xy'-x'y)+4N\Theta'^2(xy'-x'y)^2 \\
& +S\,[xy\cos 2\Theta+\tfrac{1}{2}(x^2-y^2)\sin 2\Theta]\,\frac{\mathrm{d}}{\mathrm{d}z}\,[(x^2-y^2)\cos 2\Theta-2xy\sin 2\Theta] \\
& +T[xy\cos 2\Theta+\tfrac{1}{2}(x^2-y^2)\sin 2\Theta]\,\frac{\mathrm{d}}{\mathrm{d}z}\,(x^2+y^2)
\end{aligned}
\tag{A 5c}
$$

in which

$$
\left.
\begin{aligned}
&\hat{j}=F+\tfrac{1}{2}G+K\Theta'^2+N\Theta'^4, \quad \hat{k}=\surd(G^2+\tfrac{1}{4}I^2), \\
&\hat{m}=\surd\{\tfrac{1}{4}(H+M\Theta'^2+R\Theta')^2+(J+L\Theta'^2)^2\}, \quad \hat{n}=K+2N\Theta'^2, \\
&\hat{q}=\surd(L^2+\tfrac{1}{4}M^2), \quad \hat{r}=\surd\{4L^2\Theta'^2+(M\Theta'+\tfrac{1}{2}R)^2\}, \\
&\tan\Pi=2G/I, \quad \tan\Sigma=\frac{2(J+L\Theta'^2)}{H+M\Theta'^2+R\Theta'}, \\
&\tan\Xi=2L/M, \quad \tan\Lambda=\frac{2L\Theta'}{M\Theta'+\frac{1}{2}R}.
\end{aligned}
\right\}
\tag{A 6}
$$

In orthogonal systems, $m^{(2)}$ can be cast into a form containing terms in $(x'^2+y'^2)$, x^2 and y^2 only, so that

$$
\tan 2\Theta=\left[\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}P+\eta\,\Delta\right]\Big/\left[\frac{1}{2}\,\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}D-2\eta Q\right]
\tag{A 7}
$$

which, in conjunction with the definition of Θ, is known as the *orthogonality condition*. This condition fulfilled, the Euler equations of

$$
\delta\int m^{(2)}\,\mathrm{d}z=0
$$

separate, and their general solutions can be formally written

$$
\left.
\begin{aligned}
x(z)&=x_o\,g_x(z)+x_a\,h_x(z), \\
y(z)&=y_o\,g_y(z)+y_a\,h_y(z).
\end{aligned}
\right\}
\tag{A 8}
$$

Substituting these solutions into $m^{(4)}$, we obtain

$$
\begin{aligned}
m^{(4)} = {} & \bar{a}x_o^4+\bar{b}y_o^4+\bar{c}x_a^4+\bar{d}y_a^4 \\
& +\bar{e}x_o^2y_o^2+\bar{f}x_a^2y_a^2+\bar{g}x_o^2y_a^2+\bar{h}x_a^2y_o^2+\bar{j}x_o^2x_a^2+\bar{k}y_o^2y_a^2 \\
& +\bar{l}x_o^3x_a+\bar{m}y_o^3y_a+\bar{n}x_o\,x_a^3+\bar{p}y_o\,y_a^3+\bar{q}x_o^3y_a+\bar{r}x_o^3y_o \\
& +\bar{s}y_o^3x_a+\bar{t}y_o^3x_o+\bar{u}y_o\,x_a^3+\bar{v}y_a\,x_a^3+\bar{w}x_o\,y_a^3+\bar{z}x_a\,y_a^3 \\
& +\bar{\alpha}x_o^2x_a\,y_o+\bar{\beta}\,x_o^2x_a\,y_a+\bar{\gamma}x_o^2y_o\,y_a+\bar{\delta}x_a^2x_o\,y_o \\
& +\bar{\xi}x_a^2x_o\,y_a+\bar{\zeta}x_a^2y_o\,y_a+\bar{\lambda}y_o^2x_o\,x_a+\bar{\mu}y_o^2x_o\,y_a \\
& +\bar{\nu}y_o^2x_a\,y_a+\bar{\theta}y_a^2x_o\,y_o+\bar{\phi}y_a^2x_o\,x_a+\bar{\rho}y_a^2x_a\,y_o+\bar{\omega}x_o\,y_o\,x_a\,y_a
\end{aligned}
\tag{A 9}
$$

in which

$$\bar{a} = [\hat{j}+\tfrac{1}{2}\hat{k}\sin(4\Theta+\Pi)-S\Theta'\sin^2 2\Theta+\hat{m}\sin(2\Theta+\Sigma)]\,g_x^4 + \sin 2\Theta(S\cos 2\Theta+T)\,g_x^3 g_x' + [\hat{n}+\hat{q}\sin(2\Theta+\Xi)]\,g_x g_x'^2 + N g_x'^4. \quad \text{(A 10}a\text{)}$$

Writing h_x instead of g_x converts $\bar{a}$ into $\bar{c}$; writing g_y instead of g_x, $-\hat{m}$ instead of $\hat{m}$ and $-\hat{q}$ instead of $\hat{q}$ converts $\bar{a}$ into $\bar{b}$; writing h_y instead of g_y converts $\bar{b}$ into $\bar{d}$

$$\begin{aligned}\bar{e} = {} & 2[\hat{j}-\tfrac{3}{2}\hat{k}\sin(4\Theta+\Pi)-2S\Theta'\cos^2 2\Theta+S\Theta'\sin^2 2\Theta]\,g_x^2 g_y^2 \\ & +2Ng_x'^2 g_y'^2+[\hat{n}+\hat{q}\sin(2\Theta+\Xi)]\,g_x^2 g_y'^2+[\hat{n}-\hat{q}\sin(2\Theta+\Xi)]\,g_x'^2 g_y^2 \\ & -\tfrac{3}{2}\sin 4\Theta S g_x g_y (g_x g_y)' \\ & +[2\hat{r}\cos(2\Theta+\Lambda)+T\sin 2\Theta]\,g_x g_y (g_x g_y' - g_x' g_y) \\ & +4N\Theta'^2 (g_x g_y' - g_x' g_y)^2. \end{aligned} \quad \text{(A 10}e\text{)}$$

$\bar{f}$, $\bar{g}$ and $\bar{h}$ are identical to $\bar{e}$, apart from the following exchanges:

$\bar{f}$: as $\bar{e}$ if g_x is replaced by h_x, g_y by h_y, g_x' by h_x' and g_y' by h_y',
$\bar{g}$: as $\bar{e}$ if g_y is replaced by h_y and g_y' by h_y',
$\bar{h}$: as $\bar{e}$ if g_x is replaced by h_x and g_x' by h_x',

$$\begin{aligned}\bar{j} = {} & 6[\hat{j}+\tfrac{1}{2}\hat{k}\sin(4\Theta+\Pi)-S\Theta'\sin^2 2\Theta+\hat{m}\sin(2\Theta+\Sigma)]\,g_x^2 h_x^2+6Ng_x'^2 h_x'^2 \\ & +[\hat{n}+\hat{q}\sin(2\Theta+\Sigma)]\,(g_x^2 h_x'^2+g_x'^2 h_x^2+4g_x g_x' h_x h_x') \\ & +3\sin 2\Theta(S\cos 2\Theta+T)\,g_x h_x (g_x h_x)'. \end{aligned} \quad \text{(A 10}j\text{)}$$

Writing g_y instead of g_x and h_y instead of h_x, and changing the signs of $\hat{m}$, $\hat{q}$ and T, converts $\bar{j}$ into $\bar{k}$.

$$\begin{aligned}\bar{l} = {} & 4[\hat{j}+\tfrac{1}{2}\hat{k}\sin(4\Theta+\Pi)-S\Theta'\sin^2 2\Theta+\hat{m}\sin(2\Theta+\Sigma)]\,g_x^3 h_x \\ & +4Ng_x'^3 h_x'+2[\hat{n}+\hat{q}\sin(2\Theta+\Xi)]\,g_x g_x'(g_x h_x)' \\ & +\sin 2\Theta(S\cos 2\Theta+T)\,[g_x^2(g_x h_x)'+2g_x^2 g_x' h_x]. \end{aligned} \quad \text{(A 10}l\text{)}$$

$\bar{n}$ is identical to $\bar{l}$ if $h_x(h_x')$ and $g_x(g_x')$ are interchanged.

Replacing g_x by g_y and h_x by h_y, and changing the signs of $\hat{n}$, $\hat{q}$, and T converts $\bar{l}$ into $\bar{m}$.
$\bar{p}$ is derived from $\bar{m}$ by interchanging h_y and g_y.

$$\begin{aligned}\bar{q} = {} & 2[\hat{k}\cos(4\Theta+\Pi)-S\Theta'\sin 4\Theta+\hat{m}\cos(2\Theta+\Sigma)]\,g_x^3 h_y \\ & +[2\hat{n}\Theta'+\hat{r}\sin(2\Theta+\Lambda)]\,g_x^2(g_x h_y'-g_x' h_y) \\ & +2\cos 2\Theta(T+S\cos 2\Theta)\,g_x^2 g_x' h_y \\ & -S\sin^2 2\Theta g_x^2 (g_x g_y)'+2\hat{q}\cos(2\Theta+\Xi)\,g_x g_x'^2 g_y \\ & +4N\Theta' g_x'^2 (g_x g_y'-g_x' g_y). \end{aligned} \quad \text{(A 10}q\text{)}$$

$\bar{r}$, $\bar{u}$ and $\bar{v}$ are identical to $\bar{q}$ if the following functions are exchanged:

$\bar{r}$ is derived from $\bar{q}$ by replacing h_y by g_y and h_y' by g_y';
$\bar{u}$ is derived from $\bar{q}$ by replacing g_x by h_x, h_y by g_y, g_x' by h_x' and h_y' by g_y';
$\bar{v}$ is derived from $\bar{q}$ by replacing g_x by h_x and g_x' by h_x';
$-\bar{s}$ is derived from $\bar{q}$ by replacing g_x by g_y and h_y by h_x, and changing the signs of $\hat{m}$, $\hat{r}$, T, $\hat{q}$.

$\bar{t}$, $\bar{w}$ and $\bar{z}$ are identical to $\bar{s}$ if the functions are exchanged in the following ways:

$\bar{t}$ is derived from $\bar{s}$ by replacing h_x by g_x and h'_x by g'_x;
$\bar{w}$ is derived from $\bar{s}$ by replacing h_x by g_x, g_y by h_y, h'_x by g'_x and g'_y by h'_y;
$\bar{z}$ is derived from $\bar{s}$ by replacing g_y by h_y and g'_y by h'_y;

$$\begin{aligned}\bar{\alpha} = {} & 6[\hat{k}\cos(4\Theta+\Pi)+\hat{m}\cos(2\Theta+\Sigma)-S\Theta'\sin 4\Theta]\, g_x^2 h_x g_y \\ & +[2\hat{n}\Theta'+\hat{r}\sin(2\Theta+\Lambda)]\,[g_x^2(h_x g'_y-h'_x g_y)+2g_x h_x(g_x g'_y-g'_x g_y)] \\ & +2\hat{q}\cos(2\Theta+\Xi)\,(2g_x g_y g'_x h'_x+h_x g'^2_x g_y) \\ & +4N\Theta'[g'^2_x(h_x g'_y-h'_x g_y)+2g'_x h'_x(g_x g'_y-g'_x g_y)] \\ & +2\cos 2\Theta(T+S\cos 2\Theta)\,(g_x h_x\, g'_x g_y+(g_x h_x)'\, g_x g_y) \\ & -S\sin^2 2\Theta[2g_x h_x(g_x g_y)'+g_x^2(g_y h_x)'].\end{aligned} \qquad (\mathrm{A}\,10\alpha)$$

The following exchanges convert $\bar{\alpha}$ into $\bar{\beta}$, $\bar{\delta}$ or $\bar{\xi}$:

$\bar{\beta}$ is derived from $\bar{\alpha}$ if g_y is replaced by h_y and g'_y by h'_y;
$\bar{\delta}$ is derived from $\bar{\alpha}$ if g_x is replaced by h_x, g'_x by h'_x, h_x by g_x and h'_x by g'_x;
$\bar{\xi}$ is derived from $\bar{\alpha}$ if g_x is replaced by h_x, g'_x by h'_x, g_y by h_y, g'_y by h'_y, h_x by g_x and h'_x by g'_x.

If g_x is replaced by g_y, g_y by g_x and h_x by h_y, and if the signs of $\hat{m}$, $\hat{q}$, $\hat{r}$ and T are reversed, $\bar{\alpha}$ is converted into $-\bar{\mu}$.

Just as $\bar{\alpha}$ generates $\bar{\beta}$, $\bar{\delta}$ and $\bar{\xi}$, so $\bar{\mu}$ generates $\bar{\nu}$, $\bar{\theta}$ and $\bar{\rho}$, thus:

ν is derived from $\bar{\mu}$ if g_x is replaced by h_x and g'_x by h'_x;
$\bar{\theta}$ is derived from $\bar{\mu}$ if g_y is replaced by h_y, g'_y by h'_y, h_y by g_y and h'_y by g'_y;
$\bar{\rho}$ is derived from $\bar{\mu}$ if g_x is replaced by h_x, g'_x by h'_x, g_y by h_y, g'_y by h'_y, h_y by g_y and h'_y by g'_y;

$$\begin{aligned}\bar{\gamma} = {} & 4[\hat{j}-\tfrac{3}{2}\hat{k}\sin(4\Theta+\Pi)-2S\Theta'\cos^2 2\Theta+S\Theta'\sin^2 2\Theta]\, g_x^2 g_y h_y \\ & +4Ng'^2_x g'_y h'_y+2[\hat{n}+\hat{q}\sin(2\Theta+\Xi)]\, g_x^2 g'_y h'_y \\ & +2[\hat{n}-\hat{q}\sin(2\Theta+\Xi)]\, g'^2_x g_y h_y \\ & +2\hat{r}\cos(2\Theta+\Lambda)\,[g_x g_y(g_x h'_y-g'_x h_y)+g_x h_y(g_x g'_y-g'_x g_y)] \\ & +8N\Theta'^2(g_x g'_y-g'_x g_y)\,(g_x h'_y-g'_x h_y) \\ & -(3S\sin 4\Theta+2T\sin 2\Theta)\, g_x g'_x g_y h_y \\ & -\tfrac{1}{2}(3S\sin 4\Theta-2T\sin 2\Theta)\, g_x^2(g_y h_y)'.\end{aligned} \qquad (\mathrm{A}\,10\gamma)$$

Replacing g_x by h_x converts $\bar{\gamma}$ into $\bar{\zeta}$.

$\bar{\lambda}$ is derived from $\bar{\gamma}$ by interchanging g_x and g_y, replacing h_y by h_x, and reversing the signs of $\hat{q}$, $\hat{r}$ and T.

Writing h_y instead of g_y converts $\bar{\lambda}$ into ϕ.

$$\begin{aligned}\bar{\omega} = {} & 8[\hat{j}-\tfrac{3}{2}\hat{k}\sin(4\Theta+\Pi)-2S\Theta'\cos^2 2\Theta+S\Theta'\sin^2 2\Theta]\, g_x h_x g_y h_y \\ & +8Ng'_x h'_x g'_y h'_y+4[\hat{n}+\hat{q}\sin(2\Theta+\Sigma)]\, g_x h_x g'_y h'_y \\ & +4[\hat{n}-\hat{q}\sin(2\Theta+\Sigma)]\, g'_x h'_x g_y h_y \\ & +4\hat{r}\cos(2\Theta+\Lambda)\,[g_x g_y(h_x h'_y-h'_x h_y)+h_x h_y(g_x g'_y-g'_x g_y)] \\ & +8N\Theta'^2[2(g_x g'_y h_x h'_y+g'_x g_y h'_x h_y)-(g_x h_x)'\,(g_y h_y)'] \\ & -S\sin 4\Theta(g_x g_y h_x h_y)' \\ & -4[S(g_x h_x g_y h_y)'-\tfrac{1}{2}T\{g_x h_x(g_y h_y)'-g_y h_y(g_x h_x)'\}].\end{aligned} \qquad (\mathrm{A}\,10\omega)$$

Sturrock (1951) has shown that the primary aberrations of an orthogonal system in an arbitrary current plane, $z = z_c$, are given by

$$\left.\begin{aligned} x_c^{\mathrm{I}} &= \frac{h_{xc}(\partial V_{ac}^{\mathrm{I}}/\partial x_o) - g_{xc}(\partial V_{oc}^{\mathrm{I}}/\partial x_a)}{\sqrt{\{\Phi(1+\epsilon\Phi)\}}\,(g_{xc}h'_{xc} - g'_{xc}h_{xc})}, \\ y_c^{\mathrm{I}} &= \frac{h_{yc}(\partial V_{ac}^{\mathrm{I}}/\partial y_o) - g_{yc}(V_{oc}^{\mathrm{I}}/y_a)}{\sqrt{\{\Phi(1+\epsilon\Phi)\}}\,(g_{yc}h'_{yc} - g'_{yc}h_{yc})}, \end{aligned}\right\} \tag{A 11}$$

in which the suffix 'c' appended to a function of z implies that the function is to be evaluated in the plane $z = z_c$;

$$V^{\mathrm{I}} = \int m^{(4)}\,\mathrm{d}z = x_o^4\int \bar{a}\,\mathrm{d}z + y_o^4\int \bar{b}\,\mathrm{d}z + \ldots + x_o y_o x_a y_a \int \bar{\omega}\,\mathrm{d}z$$

and we denote $\int_{z_o}^{z_c} \bar{a}\,\mathrm{d}z$ by $a^\dagger$ (A 12a)

and $\int_{z_a}^{z_c} \bar{a}\,\mathrm{d}z$ by a^* (A 12b)

with a corresponding notation for each of the other coefficient integrals.

Denoting the aberration coefficients by $(\alpha\beta\gamma\delta)_x$ and $(\alpha\beta\gamma\delta)_y$, so that

$$\begin{aligned} x_c^{\mathrm{I}} = {} & (3000)_x x_o^3 + (0300)_x y_o^3 + (0030)_x x_a^3 + (0003)_x y_a^3 \\ & + (2100)_x x_o^2 y_o + (1200)_x x_o y_o^2 + (2010)_x x_o^2 x_a + (1020)_x x_o x_a^2 \\ & + (2001)_x x_o^2 y_a + (1002)_x x_o y_a^2 + (0210)_x y_o^2 x_a + (0120)_x y_o x_a^2 \\ & + (0201)_x y_o^2 y_a + (0102)_x y_o y_a^2 + (0021)_x x_a^2 y_a + (0012)_x x_a y_a^2 \\ & + (1110)_x x_o y_o x_a + (1101)_x x_o y_o y_a + (1011)_x x_o x_a y_a + (0111)_x y_o x_a y_a \end{aligned} \tag{A 13}$$

with an identical expression for y^{I}, except that $(\alpha\beta\gamma\delta)_x$ is replaced by $(\alpha\beta\gamma\delta)_y$, we find

$$\left.\begin{aligned}
k_x(3000)_x &= 4h_x a^* - g_x l^\dagger, & k_y(3000)_y &= h_y r^* - g_y q^\dagger, \\
k_x(0300)_x &= h_x t^* - g_x s^\dagger, & k_y(0300)_y &= 4h_y b^* - g_y m^\dagger, \\
k_x(0030)_x &= h_x n^* - 4g_x c^\dagger, & k_y(0030)_y &= h_y u^* - g_y v^\dagger, \\
k_x(0003)_x &= h_x w^* - g_x z^\dagger, & k_y(0003)_y &= h_y p^* - 4g_y d^\dagger, \\
k_x(2100)_x &= 3h_x r^* - g_x \alpha^\dagger, & k_y(2100)_y &= 2h_y e^* - g_y \gamma^\dagger, \\
k_x(1200)_x &= 2h_x e^* - g_x \lambda^\dagger, & k_y(1200)_y &= 3h_y t^* - g_y \mu^\dagger, \\
k_x(2010)_x &= 3h_x l^* - 2g_x j^\dagger, & k_y(2010)_y &= h_y \alpha^* - g_y \beta^\dagger, \\
k_x(1020)_x &= 2h_x j^* - 3g_x n^\dagger, & k_y(1020)_y &= h_y \delta^* - g_y \xi^\dagger, \\
k_x(2001)_x &= 3h_x q^* - g_x \beta^\dagger, & k_y(2001)_y &= h_y \gamma^* - 2g_y g^\dagger, \\
k_x(1002)_x &= 2h_x g^* - g_x \phi^\dagger, & k_y(1002)_y &= h_y \theta^* - 3g_y w^\dagger, \\
k_x(0210)_x &= h_x \lambda^* - 2g_x h^\dagger, & k_y(0210)_y &= 3h_y s^* - g_y \nu^\dagger, \\
k_x(0120)_x &= h_x \delta^* - 3g_x u^\dagger, & k_y(0120)_y &= 2h_y h^* - g_y \zeta^\dagger, \\
k_x(0201)_x &= h_x \mu^* - g_x \nu^\dagger, & k_y(0201)_y &= 3h_y m^* - 2g_y k^\dagger, \\
k_x(0102)_x &= h_x \theta^* - g_x \rho^\dagger, & k_y(0102)_y &= 2h_y k^* - 3g_y p,^\dagger \\
k_x(0021)_x &= h_x \xi^* - 3g_x v^\dagger, & k_y(0021)_y &= h_y \zeta^* - 2g_y f^\dagger, \\
k_y(0012)_x &= h_x \phi^* - 2g_x f^\dagger, & k_y(0012)_y &= h_y \rho^* - 3g_y z^\dagger, \\
k_x(1110)_x &= 2h_x \alpha^* - 2g_x \delta^\dagger, & k_y(1110)_y &= 2h_y \lambda^* - g_y \omega^\dagger, \\
k_x(1101)_x &= 2h_x \gamma^* - g_x \omega^\dagger, & k_y(1101)_y &= 2h_y \mu^* - 2g_y \theta^\dagger, \\
k_x(1011)_x &= 2h_x \beta^* - 2g_x \xi^\dagger, & k_y(1011)_y &= h_y \omega^* - 2g_y \phi^\dagger, \\
k_x(0111)_x &= h_x \omega^* - 2g_x \zeta^\dagger, & k_y(0111)_y &= 2h_y \nu^* - 2g_y \rho^\dagger.
\end{aligned}\right\} \tag{A 14}$$

The symbols k_x and k_y denote the (constant) quantities $\surd\{\Phi(1+\epsilon\Phi)\}(g_x h_x' - g_x' h_x)$ and $\surd\{\Phi(1+\epsilon\Phi)\}(g_y h_y' - g_y' h_y)$, respectively.

In the complex notation described in part I, we have

$$u_c = \Sigma(\alpha\beta\gamma\delta)\, u_o^\alpha \bar{u}_o^\beta u_a^\gamma \bar{u}_a^\delta \tag{A 15}$$

and if we denote $1/8k_x$ by k_x' and $1/8k_y$ by k_y', the coefficients $(\alpha\beta\gamma\delta)$ are of the following form:

$$\begin{aligned}
(3000) = {} & k_x'[h_x(4a^*-2e^*)-g_x(l^\dagger-\lambda^\dagger)]+k_y'[h_y(-4b^*+2e^*)-g_y(-m^\dagger+\gamma^\dagger)]\\
&+\mathrm{i}k_x'[h_x(t^*-3r^*)-g_x(s^\dagger-\alpha^\dagger)]+\mathrm{i}k_y'[h_y(r^*-3t^*)-g_y(q^\dagger-\mu^\dagger)],\\
(0300) = {} & k_x'[h_x(4a^*-2e^*)-g_x(l^\dagger-\lambda^\dagger)]+k_y'[h_y(4b^*-2e^*)-g_y(m^\dagger-\gamma^\dagger)]\\
&+\mathrm{i}k_x'[h_x(-t^*+3r^*)-g_x(-s^\dagger+\alpha^\dagger)]+\mathrm{i}k_y'[h_y(r^*-3t^*)-g_y(q^\dagger-\mu^\dagger)],\\
(0030) = {} & k_x'[h_x(n^*-\phi^*)-g_x(4c^\dagger-2f^\dagger)]+k_y'[h_y(-p^*+\zeta^*)-g_y(-4d^\dagger+2f^\dagger)]\\
&+\mathrm{i}k_x'[h_x(w^*-\xi^*)-g_x(z^\dagger-3v^\dagger)]+\mathrm{i}k_y'[h_y(u^*-\rho^*)-g_y(v^\dagger-3z^\dagger)],\\
(0003) = {} & k_x'[h_x(n^*-\phi^*)-g_x(4c^\dagger-2f^\dagger)]+k_y'[h_y(p^*-\zeta^*)-g_y(4d^\dagger-2f^\dagger)]\\
&+\mathrm{i}k_x'[h_x(-w^*+\xi^*)-g_x(-z^\dagger+3v^\dagger)]+\mathrm{i}k_y'[h_y(u^*-\rho^*)-g_y(v^\dagger-3z^\dagger)],\\
(2100) = {} & k_x'[h_x(12a^*+2e^*)-g_x(3l^\dagger+\lambda^\dagger)]+k_y'[h_y(12b^*+2e^*)-g_y(3m^\dagger+\gamma^\dagger)]\\
&+\mathrm{i}k_x'[-h_x(3t^*+3r^*)+g_x(3s^\dagger+\alpha^\dagger)]+\mathrm{i}k_y'[h_y(3r^*+3t^*)-g_y(3q^\dagger+\mu^\dagger)],\\
(1200) = {} & k_x'[h_x(12a^*+2e^*)-g_x(3l^\dagger+\lambda^\dagger)]+k_y'[-h_y(12b^*+2e^*)+g_y(3m^\dagger+\gamma^\dagger)]\\
&+\mathrm{i}k_x'[h_x(3t^*+3r^*)-g_x(3s^\dagger+\alpha^\dagger)]+\mathrm{i}k_y'[h_y(3r^*+3t^*)-g_y(3q^\dagger+\mu^\dagger)],\\
(0021) = {} & k_x'[h_x(3t^*+\phi^*)-g_x(3s^\dagger+2f^\dagger)]+k_y'[h_y(3u^*+\zeta^*)-g_y(3v^\dagger+2f^\dagger)]\\
&+\mathrm{i}k_x'[-h_x(3w^*+\xi^*)+g_x(3z^\dagger+3v^\dagger)]+\mathrm{i}k_y'[h_y(3u^*+\rho^*)-g_y(3v^\dagger+3z^\dagger)],\\
(0012) = {} & k_x'[h_x(3n^*+\phi^*)-g_x(12c^\dagger+2f^\dagger)]+k_y'[-h_y(3p^*+\zeta^*)+g_y(12d^\dagger+2f^\dagger)]\\
&+\mathrm{i}k_x'[h_x(3w^*+\xi^*)-g_x(3z^\dagger+3v^\dagger)]+\mathrm{i}k_y'[h_y(3u^*+\rho^*)-g_y(3v^\dagger+3z^\dagger)],\\
(2010) = {} & k_x'[h_x(3l^*-\lambda^*-2\gamma^*)-g_x(2j^\dagger-2h^\dagger-\omega^\dagger)]\\
&+k_y'[h_y(\gamma^*-3m^*+2\lambda^*)-g_y(2g^\dagger-2k^\dagger+\omega^\dagger)]\\
&+\mathrm{i}k_x'[h_x(\mu^*-3q^*-2\alpha^*)-g_x(\nu^\dagger-\beta^\dagger-2\delta^\dagger)]\\
&+\mathrm{i}k_y'[h_y(\alpha^*-3s^*-2\mu^*)-g_y(\beta^\dagger-\nu^\dagger-2\theta^\dagger)],\\
(1020) = {} & k_x'[h_x(2j^*-2g^*-\omega^*)-g_x(3n^\dagger-\phi^\dagger-2\zeta^\dagger)]\\
&+k_y'[h_y(2h^*-2k^*+\omega^*)-g_y(\zeta^\dagger-3p^\dagger+2\phi^\dagger)]\\
&+\mathrm{i}k_x'[h_x(\theta^*-\delta^*-2\beta^*)-g_x(\rho^\dagger-3u^\dagger-2\xi^\dagger)]\\
&+\mathrm{i}k_y'[h_y(\delta^*-\theta^*-2\nu^*)-g_y(\xi^\dagger-3w^\dagger-2\rho^\dagger)],\\
(2001) = {} & k_x'[h_x(3l^*-\lambda^*-2\gamma^*)-g_x(2j^\dagger-2h^\dagger-\omega^\dagger)]\\
&+k_y'[h_y(3m^*-\gamma^*+2\lambda^*)-g_y(2k^\dagger-2g^\dagger+\omega^\dagger)]\\
&+\mathrm{i}k_x'[h_x(3q^*-\mu^*-2\alpha^*)-g_x(\beta^\dagger-\nu^\dagger-2\delta^\dagger)]\\
&+\mathrm{i}k_y'[h_y(\alpha^*-3s^*+2\mu^*)-g_y(\beta^\dagger-\nu^\dagger+2\theta^\dagger)],
\end{aligned}$$

$$\left.\begin{aligned}
(0120) &= k'_x[h_x(2j^*-2g^*-\omega^*)-g_x(3n^\dagger-\phi^\dagger-2\zeta^\dagger)]\\
&\quad+k'_y[h_y(2k^*-2h^*+\omega^*)-g_y(3p^\dagger-\zeta^\dagger+2\phi^\dagger)]\\
&\quad+\mathrm{i}k'_x[h_x(\delta^*-\theta^*-2\beta^*)-g_x(3u^\dagger-\rho^\dagger-2\xi^\dagger)]\\
&\quad+\mathrm{i}k'_y[h_y(\delta^*-\theta^*-2\nu^*)-g_y(\xi^\dagger-3w^\dagger-2\rho^\dagger)],\\
(0210) &= k'_x[h_x(3l^*-\lambda^*+2\gamma^*)-g_x(2j^\dagger-2h^\dagger+\omega^\dagger)]\\
&\quad+k'_y[h_y(\gamma^*-3m^*-2\lambda^*)-g_y(2g^\dagger-2k^\dagger-\omega^\dagger)]\\
&\quad+\mathrm{i}k'_x[h_x(\mu^*-3q^*+2\alpha^*)-g_x(\nu^\dagger-\beta^\dagger+2\delta^\dagger)]\\
&\quad+\mathrm{i}k'_y[h_y(\alpha^*-3s^*+2\mu^*)-g_y(\beta^\dagger-\nu^\dagger+2\theta^\dagger)],\\
(1002) &= k'_x[h_x(2j^*-2g^*+\omega^*)-g_x(3n^\dagger-\phi^\dagger+2\zeta^\dagger)]\\
&\quad+k'_y[h_y(2h^*-2k^*-\omega^*)-g_y(\zeta^\dagger-3p^\dagger-2\phi^\dagger)]\\
&\quad+\mathrm{i}k'_x[h_x(\theta^*-\delta^*+2\beta^*)-g_x(\rho^\dagger-3u^\dagger+2\xi^\dagger)]\\
&\quad+\mathrm{i}k'_y[h_y(\delta^*-\theta^*+2\nu^*)-g_y(\xi^\dagger-2w^\dagger+2\rho^\dagger)],\\
(0201) &= k'_x[h_x(3l^*-\lambda^*-2\gamma^*)-g_x(2j^\dagger-2h^\dagger-\omega^\dagger)]\\
&\quad+k'_y[h_y(3m^*-\gamma^*-2\lambda^*)-g_y(2k^\dagger-2g^\dagger-\omega^\dagger)]\\
&\quad+\mathrm{i}k'_x[h_x(3q^*-\mu^*+2\alpha^*)-g_x(\beta^\dagger-\nu^\dagger+2\delta^\dagger)]\\
&\quad+\mathrm{i}k'_y[h_y(\alpha^*-3s^*-2\mu^*)-g_y(\beta^\dagger-\nu^\dagger-2\theta^\dagger)],\\
(0102) &= k'_x[h_x(2j^*-2g^*-\omega^*)-g_x(3n^\dagger-\phi^\dagger-2\zeta^\dagger)]\\
&\quad+k'_y[h_y(2k^*-2h^*-\omega^*)-g_y(3p^\dagger-\zeta^\dagger-2\phi^\dagger)]\\
&\quad+\mathrm{i}k'_x[h_x(\delta^*-\theta^*+2\beta^*)-g_x(3u^\dagger-\rho^\dagger+2\xi^\dagger)]\\
&\quad+\mathrm{i}k'_y[h_y(\delta^*-\theta^*-2\nu^*)-g_y(\xi^\dagger-3w^\dagger-2\rho^\dagger)],\\
\tfrac{1}{2}(1110) &= k'_x[h_x(3l^*+\lambda^*)-g_x(2j^\dagger+2h^\dagger)]\\
&\quad+k'_y[h_y(\gamma^*+3m^*)-g_y(2g^\dagger+2k^\dagger)]\\
&\quad+\mathrm{i}k'_x[-h_x(3q^*+\mu^*)+g_x(\beta^\dagger+\nu^\dagger)]\\
&\quad+\mathrm{i}k'_y[h_y(\alpha^*+3s^*)-g_y(\beta^\dagger+\nu^\dagger)],\\
\tfrac{1}{2}(1011) &= k'_x[h_x(2j^*+2g^*)-g_x(n^\dagger+\phi^\dagger)]\\
&\quad+k'_y[h_y(2h^*+2k^*)-g_y(\zeta^\dagger+3p^\dagger)]\\
&\quad+\mathrm{i}k'_x[-h_x(\delta^*+\theta^*)+g_x(3u^\dagger+\rho^\dagger)]\\
&\quad+\mathrm{i}k'_y[h_y(\delta^*+\theta^*)-g_y(\xi^\dagger+3w^\dagger)],\\
\tfrac{1}{2}(1101) &= k'_x[h_x(3l^*+\lambda^*)-g_x(2j^\dagger+2h^\dagger)]\\
&\quad+k'_y[-h_y(\gamma^*+3m^*)+g_y(2g^\dagger+2k^\dagger)]\\
&\quad+\mathrm{i}k_x[h_x(3q^*+\mu^*)-g_x(\beta^\dagger+\nu^\dagger)]\\
&\quad+\mathrm{i}k'_y[h_y(\alpha^*+3s^*)-g_y(\beta^\dagger+\nu^\dagger)],\\
\tfrac{1}{2}(0111) &= k'_x[h_x(2j^*+2g^*)-g_x(3n^\dagger+\phi^\dagger)]\\
&\quad+k'_y[-h_y(2h^*+2k^*)+g_y(\zeta^\dagger+3p^\dagger)]\\
&\quad+\mathrm{i}k'_x[h_x(\delta^*+\theta^*)-g_x(3u^\dagger+\rho^\dagger)]\\
&\quad+\mathrm{i}k'_y[h_y(\delta^*+\theta^*)-g_y(\xi^\dagger+3w^\dagger)].
\end{aligned}\right\}\quad\text{(A 16)}$$

An important special case: the 'twist-free' orthogonal system‡

In practice, it is very difficult to design a rectilinear orthogonal system in which the angle Θ is not a constant, for if Θ is a function of z, a delicate balance of magnetic and electrostatic fields has to be established and maintained. The class of orthogonal systems for which Θ is constant is therefore an important one; it includes systems of quadrupole lenses, alined so that either $P(z)$ and $\Delta(z)$ vanish identically ($\tan 2\Theta = 0$, $\Theta = 0$) or so that $D(z)$ and $Q(z)$ vanish identically ($\tan 2\Theta = \infty$, $\Theta = \frac{1}{4}\pi$). This means that any electrostatic and magnetic quadrupoles must be so orientated that the planes which bisect the electrodes are inclined at 45° to the planes which bisect the pole-pieces. Setting the axes of x and y through the electrodes (and hence midway between the pole-pieces), we obtain $\Theta = 0$. Another special case, in which Θ is constant but does not vanish, is the *composite quadrupole*, a quadrupole in which the four poles create both an electrostatic and a magnetic field.

The function $\Omega(z)$ vanishes everywhere, but $\Phi(z)$ is not restricted in any way: any electrostatic quadrupoles in the system need not carry equal and opposite potentials on their two pairs of electrodes.

It is to be stressed that the orthogonality condition only restricts the functions which affect the primordial imaging properties; thus, although $P(z)$ and $\Delta(z)$ may vanish everywhere, $P_1(z)$ and $\Delta_1(z)$ need not. This means that although the imagery may be fully described by quoting the cardinal elements in the x-z and y-z planes, the primary (third-order) aberrations will not simply be those of a system with two mutually perpendicular planes of symmetry: the quadrupoles must be alined in the way we have described, but the octopoles can be given any orientation whatsoever.

The functions $F(z), G(z), \ldots, S(z), T(z)$ can now be simplified. That $H(z)$, $M(z)$, $S(z)$ and $T(z)$ vanish follows from $\Delta(z) = P(z) = 0$; if each of the electrostatic quadrupoles is excited symmetrically, $\Phi(z)$ = constant and $K(z)$ also vanishes; if the octopoles are alined according to the same rules as the quadrupoles, so that $P_1(z) = \Delta_1(z) = 0$, then $I(z)$ is everywhere zero.

Of the four angles $\Pi(z)$, $\Sigma(z)$, $\Xi(z)$ and $\Lambda(z)$,

$$\begin{aligned}
\Pi(z) &= \tan^{-1}(2G/I) \quad \text{reduces to } \tfrac{1}{2}\pi \quad \text{if } P_1(z) = \Delta_1(z) = 0,\\
\Sigma(z) &= \tfrac{1}{2}\pi \quad \text{since} \quad H(z) = 0,\\
\Xi(z) &= \tfrac{1}{2}\pi \quad \text{since} \quad P(z) = 0,\\
\Lambda(z) &= 0 \quad \text{since} \quad \mathrm{d}\Theta(z)/\mathrm{d}z = 0.
\end{aligned}$$

Instead of setting out the new forms of the remaining functions, $F(z)$, $G(z)$, $I(z)$, $J(z)$, $K(z)$, $L(z)$, $N(z)$ and $R(z)$, the combinations of these functions which repeatedly occur in the aberration expressions will be listed. These combinations are

$$F + \tfrac{1}{2}G \pm J, \quad \surd(G^2 + \tfrac{1}{4}I^2), \quad K \pm L, \quad 2G/I;$$

the functions

$$N(z) = -\tfrac{1}{8}\surd\{\Phi(1+\epsilon\Phi)\}$$

and

$$R(z) = \tfrac{1}{4}\eta \frac{\mathrm{d}Q(z)}{\mathrm{d}z}$$

are unaffected.

‡ Simpler forms of certain expressions derived here are to be found in Hawkes (1965*b*).

$$F+\tfrac{1}{2}G = \frac{1}{128}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}\Phi^{(\mathrm{iv})} - \frac{1}{128}\frac{(\Phi'')^2}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}} - \frac{1}{256}\frac{D^2}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}} \tag{A 17a}$$

which simplifies to

$$-\frac{1}{256}\frac{D^2}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}} \tag{A 17b}$$

with symmetrical excitation.

$$J = -\frac{1}{96}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}D'' + \frac{1}{64}\frac{D\Phi''}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}} + \tfrac{1}{48}\eta Q''; \tag{A 17c}$$

here the penultimate term disappears when the excitation is symmetrical.

$$K \pm L = \frac{1+2\epsilon\Phi}{16\surd\{\Phi(1+\epsilon\Phi)\}}(-\Phi'' \pm D) \tag{A 17d}$$

which reduces to

$$\pm\frac{1+2\epsilon\Phi}{16\surd\{\Phi(1+\epsilon\Phi)\}}D \tag{A 17e}$$

with symmetrical excitation.

$$\begin{aligned} G^2+\tfrac{1}{4}I^2 = {} & \frac{(1+2\epsilon\Phi)^2}{\Phi(1+\epsilon\Phi)}(D_1^2+P_1^2) + 4\eta^2(Q_1^2+\Delta_1^2) \\ & + \frac{D^4}{128^2\Phi^3(1+\epsilon\Phi)^3} - 4\eta\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}(D_1Q_1-P_1\Delta_1) \\ & - \frac{1}{64}\frac{D^2}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}}\left(\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}D_1 - 2\eta Q_1\right), \end{aligned} \tag{A 17f}$$

$$\frac{2G}{I} = \left[\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}D_1 - \frac{1}{128}\frac{D^2}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}} - 2\eta Q_1\right] \Big/ \left[\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}P_1 + 2\eta\Delta_1\right]. \tag{A 17g}$$

We also need $F+G\pm J$, which is given by

$$\begin{aligned} & \frac{1}{128}\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}\Phi^{(\mathrm{iv})} - \frac{1}{128}\frac{(\Phi''\mp D)^2}{[\Phi(1+\epsilon\Phi)]^{\frac{3}{2}}} + \frac{1+2\epsilon\Phi}{2\surd\{\Phi(1+\epsilon\Phi)\}}D_1 \\ & \qquad - \eta Q_1 \pm \frac{1}{96}\left[-\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}D'' + 2\eta Q''\right]. \end{aligned} \tag{A 17h}$$

The functions $\hat{\jmath}, \hat{k}, \ldots, \hat{q}$ and $\hat{r}$ therefore become

$$\left.\begin{aligned} \hat{\jmath} &= F+\tfrac{1}{2}G, \\ \hat{k} &= \surd(G^2+\tfrac{1}{4}I^2) \quad (\text{or } \hat{k} = G \text{ when } P_1 = \Delta_1 = 0), \\ \hat{m} &= J, \\ \hat{n} &= K \quad (\text{or } \hat{n} = 0 \text{ when } \Phi(z) = \text{constant}), \\ \hat{q} &= L, \\ \hat{r} &= \tfrac{1}{2}R. \end{aligned}\right\} \tag{A 18}$$

The integrands $\bar{a}, \bar{b}, \ldots, \bar{\phi}, \bar{\omega}$ can now be written in the following forms:

$$\bar{a} = (F+G+J)\,g_x^4 + (K+L)\,g_x^2 g_x'^2 + N g_x'^4 \tag{A 19a}$$

$$\begin{aligned} &\bar{a} \to \bar{c} \quad \text{if} \quad g_x \to h_x \\ &\bar{a} \to \bar{b} \quad \text{if} \quad g_x \to g_y, \quad J \to -J \quad \text{and} \quad L \to -L. \\ &\bar{b} \to \bar{d} \quad \text{if} \quad g_y \to h_y. \end{aligned}$$

In these expressions, the coefficients of g_x^4 and h_x^4 simplify to $(F+G+J)$, and those of g_y^4 and h_y^4 to $(F+G-J)$, even though $P_1(z)$ and $\Delta_1(z)$ do not vanish.

$$\begin{aligned}\bar{e} = {} & (2F-2G)\, g_x^2 g_y^2 \\ & +2Ng_x'^2 g_y'^2 + (K+L)\, g_x^2 g_y'^2 + (K-L)\, g_x'^2 g_y^2 \\ & + Rg_x g_y (g_x g_y' - g_x' g_y).\end{aligned} \tag{A 19e}$$

When the excitation is symmetrical, the terms in $g_x^2 g_y'^2$ and $g_x'^2 g_y^2$ combine into

$$L(g_x^2 g_y'^2 - g_x'^2 g_y^2) \quad \text{or} \quad L(g_x g_y' - g_x' g_y)\, \mathrm{d}/\mathrm{d}z(g_x g_y).$$

The functions $\bar{f}$, $\bar{g}$ and $\bar{h}$ are derived from $\bar{e}$ by making the following exchanges:

$$\begin{aligned} \bar{e} \to \bar{f} \quad & \text{if} \quad g_x \to h_x \quad \text{and} \quad g_y \to h_y, \\ \bar{e} \to \bar{g} \quad & \text{if} \quad g_y \to h_y, \\ \bar{e} \to \bar{h} \quad & \text{if} \quad g_x \to h_x, \end{aligned}$$

$$\bar{j} = (6F+6G+6J)\, g_x^2 h_x^2 + 6Ng_x'^2 h_x'^2 + (K+L)\, (g_x^2 h_x'^2 + g_x'^2 h_x^2 + 4g_x g_x' h_x h_x'), \tag{A 19j}$$

$$\bar{l} = (4F+4G+4J)\, g_x^3 h_x + 4Ng_x'^3 h_x' + 2(K+L)\, g_x g_x' (g_x h_x)'. \tag{A 19l}$$

If $g_x \to g_y$, $h_x \to h_y$, $J \to -J$ and $L \to -L$, then $\bar{j} \to \bar{k}$ and $\bar{l} \to \bar{m}$.

$$\begin{aligned} \bar{l} \to \bar{n} \quad & \text{if} \quad h_x \to g_x \quad \text{and} \quad g_x \to h_x \\ \bar{m} \to \bar{p} \quad & \text{if} \quad h_y \to g_y \quad \text{and} \quad g_y \to h_y. \end{aligned}$$

For the skew octopole terms, we find:

$$\left.\begin{array}{llll} \bar{q} = Ig_x^3 h_y, & \bar{r} = Ig_x^3 g_y, & \bar{u} = Ih_x^3 g_y, & \bar{v} = Ih_x^3 h_y, \\ \bar{s} = -Ig_y^3 h_x, & \bar{t} = -Ig_y^3 g_x, & \bar{w} = -Ih_y^3 g_x, & \bar{z} = -Ih_y^3 h_x, \\ \bar{\alpha} = 3Ig_x^2 h_x g_y, & \bar{\beta} = 3Ig_x^2 h_x h_y, & \bar{\delta} = 3Ih_x^2 g_x g_y, & \bar{\xi} = 3Ih_x^2 g_x h_y, \\ \bar{\mu} = -3Ig_x g_y^2 h_y, & \bar{\nu} = -3Ih_x g_y^2 h_y, & \bar{\theta} = -3Ig_x h_y^2 g_y, & \bar{\rho} = -3Ih_x h_y^2 g_y. \end{array}\right\} \tag{A 19 skew octo.}$$

These sixteen expressions, $\bar{q}$, $\bar{r}$, $\bar{s}$, $\bar{t}$, $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{z}$, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\delta}$, $\bar{\xi}$, $\bar{\mu}$, $\bar{\nu}$, $\bar{\theta}$ and $\bar{\rho}$ all vanish‡ when $P_1 = \Delta_1 \equiv 0$.

$$\begin{aligned}\bar{\gamma} = {} & (4F-4G)\, g_x^2 g_y h_y \\ & + 4Ng_x'^2 g_y' h_y' + 2(K+L)\, g_x^2 g_y' h_y' + 2(K-L)\, g_x'^2 g_y h_y \\ & + R[g_x g_y (g_x h_y' - g_x' h_y) + g_x h_y (g_x g_y' - g_x' g_y)],\end{aligned} \tag{A 19γ}$$

$$\bar{\gamma} \to \bar{\zeta} \quad \text{when} \quad g_x \to h_x$$

$$\bar{\gamma} \to \bar{\lambda} \quad \text{when} \quad g_x \to g_y, \quad g_y \to g_x, \quad h_y \to h_x, \quad L \to -L \quad \text{and} \quad R \to -R,$$

$$\bar{\lambda} \to \bar{\phi} \quad \text{when} \quad g_y \to h_y.$$

‡ This result corresponds to the statement in part I of the present paper that the coefficients $(\alpha\beta\gamma\delta)$ in $u_c = \Sigma(\alpha\beta\gamma\delta)\, u_o^\alpha \bar{u}_o^\beta u_a^\gamma \bar{u}_a^\delta$ are all real in an orthogonal system with a plane of symmetry.

$$\begin{aligned}\bar{\omega} = {} & (8F-8G)\, g_x h_x g_y h_y \\ & +8N g_x' h_x' g_y' h_y' + 4(K+L)\, g_x h_x g_y' h_y' + 4(K-L)\, g_x' h_x' g_y h_y \\ & +2R[g_x g_y (h_x h_y' - h_x' h_y) + h_x h_y (g_x g_y' - g_x' g_y)].\end{aligned} \qquad \text{(A 19}\,\omega)$$

If the octopoles are alined in such a way that $P_1(z)$ and $\Delta_1(z)$ vanish, then

$$\begin{aligned}(0300)_x &= (0003)_x = (2100)_x = (2001)_x \\ &= (0120)_x = (0021)_x = (0201)_x \\ &= (0102)_x = (1110)_x = (1011)_x = 0,\end{aligned} \qquad \text{(A 20}\,a)$$

and

$$\begin{aligned}(3000)_y &= (0030)_y = (1200)_y = (2010)_y \\ &= (1020)_y = (1002)_y = (0210)_y \\ &= (0012)_y = (1101)_y = (0111)_y = 0.\end{aligned} \qquad \text{(A 20}\,b)$$

In the image plane, $z = z_i$, of a stigmatic system, the functions $h_x(z)$ and $h_y(z)$ pass through zero simultaneously, and only the contributions in $g_x(z_i) = M_x'$ and $g_y(z_i) = M_y'$ remain; M_x' and M_y' signify the magnifications in the x and y directions, respectively. In these conditions

$$\left.\begin{aligned}
(3000)_x &= -M_x l^\dagger, & (0300)_y &= -M_y m^\dagger, \\
(0030)_x &= -4M_x c^\dagger, & (0003)_y &= -4M_y d^\dagger, \\
(1200)_x &= -M_x \lambda^\dagger, & (2100)_y &= -M_y \gamma^\dagger, \\
(2010)_x &= -2M_x j^\dagger, & (0201)_y &= -2M_y k^\dagger, \\
(1020)_x &= -3M_x n^\dagger, & (0102)_y &= -3M_y p^\dagger, \\
(1002)_x &= -M_x \phi^\dagger, & (0120)_y &= -M_y \zeta^\dagger, \\
(0210)_x &= -2M_x h^\dagger, & (2001)_y &= -2M_y g^\dagger, \\
(0012)_x &= -2M_x f^\dagger, & (0021)_y &= -2M_y f^\dagger, \\
(1101)_x &= -M_x \omega^\dagger, & (1110)_y &= -M_y \omega^\dagger, \\
(0111)_x &= -2M_x \zeta^\dagger, & (1011)_y &= -2M_y \phi^\dagger,
\end{aligned}\right\} \qquad \text{(A 21)}$$

in which $M_x = M_x'/k_x$ and $M_y = M_y'/k_y$.

From these expressions, we can verify directly that if $M_x = M_y$, then

$$(0012)_x = (0021)_y$$

which is the familiar relation between the aperture aberration coefficients of a stigmatic, orthomorphic quadrupole system.‡

Similarly,

$$2(1002)_x = (1011)_y,$$

$$(0111)_x = 2(0120)_y,$$

and

$$(1101)_x = (1110)_y.$$

‡ Cf. part I, §3·2.

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To calculate the aberrations of a given system, therefore, what data are required, and how are they to be transformed into numerical values of the aberration coefficients? First, we must ascertain, from the symmetry of the system, which of the functions $\Phi(z), D(z)\ldots Q_1(z)$ appearing in equations (A 1 *a* and *b*) are present; the form of these functions must then be determined, either by measurement or computation. The functions $F, G\ldots T$ appearing in equation (A 3), and hence the functions $\hat{j}.\hat{k}\ldots\tan\Lambda$ appearing in equations (A 6), are then calculated; the components of the refractive index, $m^{(0)}$, $m^{(2)}$ and $m^{(4)}$ (A 5) thus contain known functions of z.

The paraxial equations of motion must now be solved, analytically when this is feasible, or with the aid of a computer; from the general solutions, the pairs of independent solutions g_x, h_x and g_y, h_y are constructed, which satisfy the boundary conditions

$$g_x(z_o) = g_y(z_o) = h_x(z_a) = h_y(z_a) = 1,$$
$$g_x(z_a) = g_y(z_a) = h_x(z_o) = h_y(z_o) = 0.$$

Using equation (A 8), the coefficients $\bar{a}, \bar{b}\ldots\bar{\omega}$ which appear in equation (A 9) can be calculated from their definitions (A 10), and subsequently integrated (A 12*a* and *b*). Inserting these quantities into equation (A 14), we obtain the aberration coefficients in any desired current plane.

When the system is twist-free, we calculate the combinations (A 17*a* to *h*), and instead of equations (A 10), we calculate $\bar{a}, \bar{b}\ldots\bar{\omega}$ from the simpler definitions (A 19). Otherwise, the practical calculation of the aberration coefficients follows the same course.

B. The secondary aberrations of round lens systems

In the function m, we have now to retain terms of the sixth degree in x, y, x' and y', so that

$$m = m^{(0)} + m^{(2)} + m^{(4)} + m^{(6)}.$$

Following Sturrock (1951),‡ we define the 'first form of the second-order perturbation characteristic function', $V^{\mathrm{II}}_{\alpha\beta}$, to be

$$V^{\mathrm{II}}_{\alpha\beta} = \int_{z_\alpha}^{z_\beta} \{m^{(6)}(x, y, x', y', z) - m^{(2)}(x^{\mathrm{I}}, y^{\mathrm{I}}, x'^{\mathrm{I}}, y'^{\mathrm{I}}, z)\}\,dz \tag{B 1}$$

whence we deduce that

$$\left.\begin{aligned} k_x x^{\mathrm{II}}_c &= V_3 h_{xc} - V_1 g_{xc}, \\ k_y y^{\mathrm{II}}_c &= V_4 h_{yc} - V_2 g_{yc} \end{aligned}\right\} \tag{B 2a}$$

in which

$$\left.\begin{aligned} V_1 &= \frac{\partial V^{\mathrm{II}}_{oc}}{\partial x_a} + x^{\mathrm{I}}_c \frac{\partial n^{\mathrm{I}}_{xc}}{\partial x_a} + y^{\mathrm{I}}_c \frac{\partial n^{\mathrm{I}}_{yc}}{\partial x_a}, \\ V_2 &= \frac{\partial V^{\mathrm{II}}_{oc}}{\partial y_a} + x^{\mathrm{I}}_c \frac{\partial n^{\mathrm{I}}_{xc}}{\partial y_a} + y^{\mathrm{I}}_c \frac{\partial n^{\mathrm{I}}_{yc}}{\partial y_a}, \\ V_3 &= \frac{\partial V^{\mathrm{II}}_{ac}}{\partial x_o} + x^{\mathrm{I}}_c \frac{\partial n^{\mathrm{I}}_{xc}}{\partial x_o} + y^{\mathrm{I}}_c \frac{\partial n^{\mathrm{I}}_{yc}}{\partial x_o}, \\ V_4 &= \frac{\partial V^{\mathrm{II}}_{ac}}{\partial y_o} + x^{\mathrm{I}}_c \frac{\partial n^{\mathrm{I}}_{xc}}{\partial y_o} + y^{\mathrm{I}}_c \frac{\partial n^{\mathrm{I}}_{yc}}{\partial y_o}, \end{aligned}\right\} \tag{B 2b}$$

‡ In this article, Sturrock denotes the function we call $V^{\mathrm{II}}_{\alpha\beta}$ by $*V^{\mathrm{II}}_{\alpha\beta}$.

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or in terms of known quantities

$$\left.\begin{aligned}
V_1 &= \frac{\partial V^{\mathrm{II}}_{oc}}{\partial x_a} + \frac{\surd\{\Phi(1+\epsilon\Phi)\}}{k_x^2}\left(h_x\frac{\partial V^{\mathrm{I}}_{ac}}{\partial x_o} - g_x\frac{\partial V^{\mathrm{I}}_{oc}}{\partial x_a}\right)\left(h'_x\frac{\partial^2 V^{\mathrm{I}}_{ac}}{\partial x_o\,\partial x_a} - g'_x\frac{\partial^2 V^{\mathrm{I}}_{oc}}{\partial x_a^2}\right)\\
&\quad + \frac{\surd\{\Phi(1+\epsilon\Phi)\}}{k_y^2}\left(h_y\frac{\partial V^{\mathrm{I}}_{ac}}{\partial y_o} - g_y\frac{\partial V^{\mathrm{I}}_{oc}}{\partial y_a}\right)\left(h'_y\frac{\partial^2 V^{\mathrm{I}}_{ac}}{\partial y_o\,\partial x_a} - g'_c\frac{\partial^2 V^{\mathrm{I}}_{oc}}{\partial y_a\,\partial x_a}\right),\\
V_2 &= \frac{\partial V^{\mathrm{II}}_{oc}}{\partial y_a} + \frac{\surd\{\Phi(1+\epsilon\Phi)\}}{k_x^2}\left(h_x\frac{\partial V^{\mathrm{I}}_{ac}}{\partial x_o} - g_x\frac{\partial V^{\mathrm{I}}_{oc}}{\partial x_a}\right)\left(h'_x\frac{\partial^2 V^{\mathrm{I}}_{ac}}{\partial x_o\,\partial y_a} - g'_x\frac{\partial^2 V^{\mathrm{I}}_{oc}}{\partial x_a\,\partial y_a}\right)\\
&\quad + \frac{\surd\{\Phi(1+\epsilon\Phi)\}}{k_y^2}\left(h_y\frac{\partial V^{\mathrm{I}}_{ac}}{\partial y_o} - g_y\frac{\partial V^{\mathrm{I}}_{oc}}{\partial y_a}\right)\left(h'_y\frac{\partial^2 V^{\mathrm{I}}_{ac}}{\partial y_o\,\partial y_a} - g'_y\frac{\partial^2 V^{\mathrm{I}}_{oc}}{\partial y_a^2}\right),\\
V_3 &= \frac{\partial V^{\mathrm{II}}_{ac}}{\partial x_o} + \frac{\surd\{\Phi(1+\epsilon\Phi)\}}{k_x^2}\left(h_x\frac{\partial V^{\mathrm{I}}_{ac}}{\partial x_o} - g_x\frac{\partial V^{\mathrm{I}}_{oc}}{\partial x_a}\right)\left(h'_x\frac{\partial^2 V^{\mathrm{I}}_{ac}}{\partial x_o^2} - g'_x\frac{\partial^2 V^{\mathrm{I}}_{oc}}{\partial x_a\,\partial x_o}\right)\\
&\quad + \frac{\surd\{\Phi(1+\epsilon\Phi)\}}{k_y^2}\left(h_y\frac{\partial V^{\mathrm{I}}_{ac}}{\partial y_o} - g_y\frac{\partial V^{\mathrm{I}}_{oc}}{\partial y_a}\right)\left(h'_y\frac{\partial^2 V^{\mathrm{I}}_{ac}}{\partial y_o\,\partial x_o} - g'_y\frac{\partial^2 V^{\mathrm{I}}_{oc}}{\partial y_a\,\partial x_o}\right),\\
V_4 &= \frac{\partial V^{\mathrm{II}}_{ac}}{\partial y_o} + \frac{\surd\{\Phi(1+\epsilon\Phi)\}}{k_x^2}\left(h_x\frac{\partial V^{\mathrm{I}}_{ac}}{\partial x_o} - g_x\frac{\partial V^{\mathrm{I}}_{oc}}{\partial x_a}\right)\left(h'_x\frac{\partial^2 V^{\mathrm{I}}_{ac}}{\partial x_o\,\partial y_o} - g'_x\frac{\partial^2 V^{\mathrm{I}}_{oc}}{\partial x_a\,\partial y_o}\right)\\
&\quad + \frac{\surd\{\Phi(1+\epsilon\Phi)\}}{k_y^2}\left(h_y\frac{\partial V^{\mathrm{I}}_{ac}}{\partial y_o} - g_y\frac{\partial V^{\mathrm{I}}_{oc}}{\partial y_a}\right)\left(h'_y\frac{\partial^2 V^{\mathrm{I}}_{ac}}{\partial y_o^2} - g'_y\frac{\partial^2 V^{\mathrm{I}}_{oc}}{\partial y_a\,\partial y_o}\right).
\end{aligned}\right\}\tag{B 2c}$$

The function $m^{(6)}$ is of the form

$$\begin{aligned}
m^{(6)} = m_{60}(X^2+Y^2)^3 &+ m_{42}(X^2+Y^2)^2\,(X'^2+Y'^2)\\
&+ m_{24}(X^2+Y^2)\,(X'^2+Y'^2)^2 + m_{06}(X'^2+Y'^2)^3\\
&+ m_{\Omega}(X^2+Y^2)^2\,(XY'-X'Y)
\end{aligned}\tag{B 3}$$

in which

$$\left.\begin{aligned}
m_{60} &= \tfrac{1}{1024}\surd\Phi\left[-\frac{2}{9}\frac{\Phi^{(\mathrm{vi})}}{\Phi} + \frac{\Phi^{(\mathrm{iv})}\Phi''}{\Phi^2} - \left(\frac{\Phi''}{\Phi}\right)^3\right],\\
m_{42} &= \tfrac{1}{256}\surd\Phi\left[\frac{\Phi^{(\mathrm{iv})}}{\Phi} - \left(\frac{\Phi''}{\Phi}\right)^2\right],\\
m_{24} &= \frac{1}{64}\frac{\Phi''}{\surd\Phi}, \quad m_{06} = \tfrac{1}{16}\surd\Phi
\end{aligned}\right\}\tag{B 4}$$

and
$$m_{\Omega} = -\tfrac{1}{384}\eta\Omega^{(\mathrm{iv})}.$$

On rotating the co-ordinates through the angle Θ, where

$$\frac{\mathrm{d}\Theta}{\mathrm{d}z} = \tfrac{1}{2}\eta\frac{\Omega}{\surd\Phi}\tag{B 5}$$

$m^{(6)}$ is converted into the expression

$$\begin{aligned}
m^{(6)} = m_{300}(x^2+y^2)^3 &+ m_{210}(x^2+y^2)^2\,(x'^2+y'^2)\\
&+ m_{120}(x^2+y^2)\,(x'^2+y'^2)^2 + m_{030}(x'^2+y'^2)^3\\
&+ m_{201}(x^2+y^2)^2\,(xy'-x'y) + m_{102}(x^2+y^2)\,(xy'-x'y)^2\\
&+ m_{003}(xy'-x'y)^3 + m_{021}(x'^2+y'^2)^2\,(xy'-x'y)\\
&+ m_{012}(x'^2+y'^2)\,(xy'-x'y)^2 + m_{111}(x^2+y^2)\,(x'^2+y'^2)\,(xy'-x'y)
\end{aligned}\tag{B 6}$$

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in which

$$\left.\begin{aligned} m_{300} &= m_{60}+m_{42}\,\Theta'^2+m_{24}\,\Theta'^4+m_{06}\,\Theta'^6+m_\Omega\,\Theta',\\ m_{210} &= m_{42}+2m_{24}\,\Theta'^2+3m_{06}\,\Theta'^4,\\ m_{120} &= m_{24}+3m_{06}\,\Theta'^2, \qquad m_{030} = m_{06},\\ m_{201} &= 2m_{42}\,\Theta'+4m_{24}\,\Theta'^3+6m_{06}\,\Theta'^5+m_\Omega\\ m_{102} &= 4m_{24}\,\Theta'^2+12m_{06}\,\Theta'^4,\\ m_{003} &= 8m_{06}\,\Theta'^3, \qquad m_{021} = 6m_{06}\,\Theta',\\ m_{012} &= 12m_{06}\,\Theta'^2, \quad m_{111} = 4m_{24}\,\Theta'+12m_{06}\,\Theta'^3. \end{aligned}\right\} \quad \text{(B 7)}$$

Into this expression for $m^{(6)}$, we substitute the primordial solutions,

$$\left.\begin{aligned} x &= x_o g(z)+x_a h(z),\\ y &= y_o g(z)+y_a h(z) \end{aligned}\right\} \quad \text{(B 8)}$$

to give

$$\begin{aligned} m^{(6)} = {} & \bar{a}_6(x_o^2+y_o^2)^3+\bar{b}_6(x_o^2+y_o^2)^2\,(x_a^2+y_a^2)\\ & +\bar{c}_6(x_o^2+y_o^2)\,(x_a^2+y_a^2)^2+\bar{d}_6(x_a^2+y_a^2)^3\\ & +\bar{e}_6(x_o^2+y_o^2)^2\,(x_ox_a+y_oy_a)+\bar{f}_6(x_o^2+y_o^2)\,(x_ox_a+y_oy_a)^2\\ & +\bar{j}_6(x_ox_a+y_oy_a)^3+\bar{m}_6(x_a^2+y_a^2)^2\,(x_ox_a+y_oy_a)\\ & +\bar{n}_6(x_a^2+y_a^2)\,(x_ox_a+y_oy_a)^2+\bar{p}_6(x_o^2+y_o^2)^2\,(x_oy_a-x_ay_o)\\ & +\bar{q}_6(x_o^2+y_o^2)\,(x_oy_a-x_ay_o)^2+r_6(x_oy_a-x_ay_o)^3\\ & +\bar{s}_6(x_a^2+y_a^2)^2\,(x_oy_a-x_ay_o)+\bar{t}_6(x_a^2+y_a^2)\,(x_oy_a-x_ay_o)^2\\ & +\bar{u}_6(x_ox_a+y_oy_a)^2\,(x_oy_a-x_ay_o)+\bar{v}_6(x_ox_a+y_oy_a)\,(x_oy_a-x_ay_o)^2\\ & +\bar{\lambda}_6(x_o^2+y_o^2)\,(x_a^2+y_a^2)\,(x_ox_a+y_oy_a)+\bar{\mu}_6(x_o^2+y_o^2)\,(x_a^2+y_a^2)\,(x_oy_a-x_ay_o)\\ & +\bar{\xi}_6(x_o^2+y_o^2)\,(x_ox_a+y_oy_a)\,(x_oy_a-x_ay_o)+\bar{\zeta}_6(x_a^2+y_a^2)\,(x_ox_a+y_oy_a)(x_oy_a-x_ay_o), \end{aligned} \quad \text{(B 9)}$$

where $\bar{a}_6, \bar{b}_6, \ldots, \bar{\xi}_6$ and $\bar{\zeta}_6$ are defined as follows:

$$\left.\begin{aligned} \bar{a}_6 &= m_{300}\,g^6+m_{030}\,g'^6+m_{210}\,g^4g'^2+m_{120}\,g^2g'^4,\\ \bar{d}_6 &= m_{300}\,h^6+m_{030}\,h'^6+m_{210}\,h^4h'^2+m_{120}\,h^2h'^4,\\ \bar{b}_6 &= 3m_{300}\,g^4h^2+3m_{030}\,g'^4h'^2+m_{210}\,g^2(g^2h'^2+2g'^2h^2)+m_{120}\,g'^2(g'^2h^2+2g^2h'^2),\\ \bar{c}_6 &= 3m_{300}\,g^2h^4+3m_{030}\,g'^2h'^4+m_{210}\,h^2(g'^2h^2+2g^2h'^2)+m_{120}\,h'^2(g^2h'^2+2g'^2h^2),\\ \bar{e}_6 &= 6m_{300}\,g^5h+6m_{030}\,g'^5h'+2m_{210}\,g^3g'(gh'+2g'h)+2m_{120}\,g'^3g(g'h+2gh'),\\ \bar{m}_6 &= 6m_{300}\,gh^5+6m_{030}\,g'h'^5+2m_{210}\,h^3h'(g'h+2gh')+2m_{120}\,hh'^3(gh'+2g'h),\\ \bar{f}_6 &= 12m_{300}g^4h^2+12m_{030}\,g'^4h'^2+4m_{210}\,g^2g'h(g'h+2gh')+4m_{120}\,gg'^2h'(gh'+2g'h),\\ \bar{n}_6 &= 12m_{300}\,g^2h^4+12m_{030}\,g'^2h'^4+4m_{210}\,gh^2h'(gh'+2g'h)+4m_{120}\,g'hh'^2(g'h+2gh'),\\ \bar{j}_6 &= 8[m_{300}\,g^3h^3+m_{030}\,g'^3h'^3+(m_{210}\,gh+m_{120}\,g'h')\,ghg'h']\\ \bar{r}_6 &= (gh'-g'h)^3\,m_{003},\\ \bar{q}_6 &= (m_{102}\,g^2+m_{012}\,g'^2)\,(gh'-g'h)^2,\\ \bar{t}_6 &= (m_{102}h^2+m_{012}\,h'^2)\,(gh'-g'h)^2,\\ \bar{v}_6 &= 2(m_{102}\,gh+m_{012}\,g'h')\,(gh'-g'h)^2,\\ \bar{p}_6 &= (m_{201}\,g^4+m_{021}\,g'^4+m_{111}\,g^2g'^2)\,(gh'-g'h), \end{aligned}\right\} \quad \text{(B 10)}$$

$$\left.\begin{aligned}
\bar{s}_6 &= (m_{201}h^4+m_{021}h'^4+m_{111}h^2h'^2)\,(gh'-g'h),\\
\bar{u}_6 &= [4m_{201}g^2h^2+4m_{021}g'^2h'^2+m_{111}(g^2)'\,(h^2)']\,(gh'-g'h),\\
\bar{\lambda}_6 &= 12(m_{300}g^3h^3+m_{030}g'^3h'^3)+(m_{210}gh+m_{120}g'h')\,[(g^2)'\,(h^2)'+4g^2h'^2+4g'^2h^2],\\
\bar{\mu}_6 &= [2m_{201}g^2h^2+2m_{021}g'^2h'^2+m_{111}(g^2h'^2+g'^2h^2)]\,(gh'-g'h),\\
\bar{\xi}_6 &= [4m_{201}g^3h+4m_{021}g'^3h'+m_{111}(g^2)'\,(gh)']\,(gh'-g'h),\\
\bar{\zeta}_6 &= [4m_{201}gh^3+4m_{021}g'h'^3+m_{111}(h^2)'\,(gh)']\,(gh'-g'h).
\end{aligned}\right\}$$

Our next task is to evaluate the function

$$m^{(2)}\,(x^{\mathrm{I}},y^{\mathrm{I}},x'^{\mathrm{I}},y'^{\mathrm{I}},z) \tag{B 11a}$$

given that

$$m^{(2)}\,(\xi,\eta,\xi',\eta',z) = \tfrac{1}{2}\Lambda(\xi^2+\eta^2)+\tfrac{1}{2}\Lambda'(\xi'^2+\eta'^2), \tag{B 11b}$$

where

$$\left.\begin{aligned}
\Lambda &= -\frac{1}{4}\,\frac{1+2\epsilon\Phi}{\surd\{\Phi(1+\epsilon\Phi)\}}\,\Phi''+2\surd\{\Phi(1+\epsilon\Phi)\}\,\Theta'^2,\\
\Lambda' &= \surd\{\Phi(1+\epsilon\Phi)\}
\end{aligned}\right\} \tag{B 11c}$$

and

$$\left.\begin{aligned}
kx^{\mathrm{I}} &= x_o(J^\dagger r_o^2+K^\dagger r_a^2+L^\dagger\mathbf{r}_o\,.\,\mathbf{r}_a+M^\dagger\mathbf{r}_o\times\mathbf{r}_a)+y_og(e^\dagger r_o^2+f^\dagger r_a^2+c^\dagger\mathbf{r}_o\,.\,\mathbf{r}_a)\\
&\quad+x_a(J^*r_o^2+K^*r_a^2+L^*\mathbf{r}_o\,.\,\mathbf{r}_a+M^*\mathbf{r}_o\times\mathbf{r}_a)+y_ah(e^*r_o^2+f^*r_a^2+c^*\mathbf{r}_o\,.\,\mathbf{r}_a),\\
ky^{\mathrm{I}} &= -x_og(e^\dagger r_o^2+f^\dagger r_a^2+c^\dagger\mathbf{r}_o\,.\,\mathbf{r}_a)+y_o(J^\dagger r_o^2+K^\dagger r_a^2+L^\dagger\mathbf{r}_o\,.\,\mathbf{r}_a+M^\dagger\mathbf{r}_o\times\mathbf{r}_a)\\
&\quad+x_ah(e^*r_o^2+f^*r_a^2+c^*\mathbf{r}_o\,.\,\mathbf{r}_a)+y_a(J^*r_o^2+K^*r_a^2+L^*\mathbf{r}_o\,.\,\mathbf{r}_a+M^*\mathbf{r}_o\times\mathbf{r}_a).
\end{aligned}\right\} \tag{B 12}$$

In these expressions, r_o^2 denotes $x_o^2+y_o^2$, $r_a^2=x_a^2+y_a^2$, $\mathbf{r}_o\,.\,\mathbf{r}_a=x_ox_a+y_oy_a$ and $\mathbf{r}_o\times\mathbf{r}_a=x_oy_a-x_ay_o$;

$$\left.\begin{aligned}
J^\dagger &= g\int_o^c(\mathscr{L}g^3h+\mathscr{M}gg'(gh)'+\mathscr{N}g'^3h')\,\mathrm{d}z\\
&\quad-h\int_a^c(\mathscr{L}g^4+2\mathscr{M}g^2g'^2+\mathscr{N}g'^4)\,\mathrm{d}z,\\
K^\dagger &= g\int_o^c(\mathscr{L}gh^3+\mathscr{M}hh'(gh)'+\mathscr{N}g'h'^3)\,\mathrm{d}z\\
&\quad-h\int_a^c(\mathscr{L}g^2h^2+\mathscr{M}(g^2h'^2+g'^2h^2)+\mathscr{N}g'^2h'^2+2\mathscr{K})\,\mathrm{d}z,\\
L^\dagger &= 2g\int_o^c(\mathscr{L}g^2h^2+2\mathscr{M}ghg'h'+\mathscr{N}g'^2h'^2-\mathscr{K})\,\mathrm{d}z\\
&\quad-2h\int_a^c(\mathscr{L}g^3h+\mathscr{M}gg'(gh)'+\mathscr{N}g'^3h')\,\mathrm{d}z,\\
M^\dagger &= \frac{2g}{k}\int_o^c(\mathscr{P}gh+\mathscr{Q}g'h')\,\mathrm{d}z,\\
&\quad-\frac{2h}{k}\int_a^c(\mathscr{P}g^2+\mathscr{Q}g'^2)\,\mathrm{d}z,
\end{aligned}\right\} \tag{B 13a}$$

$$\left.\begin{aligned}
J^* &= g\int_o^c (\mathscr{L}g^2h^2+\mathscr{M}(g^2h'^2+g'^2h^2)+\mathscr{N}g'^2h'^2+2\mathscr{K})\,\mathrm{d}z\\
&\quad -h\int_a^c (\mathscr{L}g^3h+\mathscr{M}gg'(gh)'+\mathscr{N}g'^3h')\,\mathrm{d}z,\\
K^* &= g\int_o^c (\mathscr{L}h^4+2\mathscr{M}h^2h'^2+\mathscr{N}h'^4)\,\mathrm{d}z\\
&\quad -h\int_a^c (\mathscr{L}gh^3+\mathscr{M}hh'(gh)'+\mathscr{N}g'h'^3)\,\mathrm{d}z,\\
L^* &= 2g\int_o^c (\mathscr{L}gh^3+\mathscr{M}hh'(gh)'+\mathscr{N}g'h'^3)\,\mathrm{d}z\\
&\quad -2h\int_a^c (\mathscr{L}g^2h^2+2\mathscr{M}ghg'h'+\mathscr{N}g'^2h'^2-\mathscr{K})\,\mathrm{d}z,\\
M^* &= \frac{2g}{k}\int_o^c (\mathscr{P}h^2+\mathscr{Q}h'^2)\,\mathrm{d}z-\frac{2h}{k}\int (\mathscr{P}gh+\mathscr{Q}g'h')\,\mathrm{d}z
\end{aligned}\right\} \qquad (\mathrm{B}\,13\,b)$$

in which the functions $\mathscr{L}(z)$, $\mathscr{M}(z)$, $\mathscr{P}(z)$, $\mathscr{Q}(z)$ and $\mathscr{K}(z)$ denote combinations of the axial potentials and fields and their derivatives, thus:

$$\left.\begin{aligned}
\mathscr{L} &= \frac{1}{32\sqrt{\{\Phi(1+\epsilon\Phi)\}}}\left[\frac{1}{\Phi(1+\epsilon\Phi)}(\Phi''+\eta^2\Omega^2)^2-\Phi^{(\mathrm{iv})}-4\eta\Omega\Omega''\right],\\
\mathscr{M} &= \frac{1}{8\sqrt{\{\Phi(1+\epsilon\Phi)\}}}[(1+2\epsilon\Phi)\Phi''+\eta^2\Omega^2],\\
\mathscr{N} &= \tfrac{1}{2}\sqrt{\{\Phi(1+\epsilon\Phi)\}},\\
\mathscr{P} &= \frac{1}{16}\frac{\eta}{\sqrt{\{\Phi(1+\epsilon\Phi)\}}}\left\{\frac{\Omega}{\Phi(1+\epsilon\Phi)}[(1+2\epsilon\Phi)\Phi''+\eta^2\Omega^2]-\Omega''\right\},\\
\mathscr{Q} &= \frac{1}{4}\eta\frac{\Omega}{\sqrt{\{\Phi(1+\epsilon\Phi)\}}},\\
\mathscr{K} &= \frac{\eta^2\Omega^2(gh'-g'h)^2}{8\sqrt{\{\Phi(1+\epsilon\Phi)\}}}.
\end{aligned}\right\} \qquad (\mathrm{B}\,14)$$

After obtaining an expression for $m^{(2)}$, we obtain expressions for kx_c^{II}, and ky_c^{II}, in the Gaussian image plane, from the formulae $kx_c^{\mathrm{II}}=-V_1g_c$, $ky_c^{\mathrm{II}}=-V_2g_c$. The coefficients of the various terms of which x_c^{II} is composed are given below;‡ in this list, the integral

$$-(g/k)\int\{\Lambda(Z_1g^2+Z_2gh+Z_3h^2)+\Lambda'(Z_1g'^2+Z_2g'h'+Z_3h'^2)\}\,\mathrm{d}z$$

is denoted concisely by $\langle\Lambda;\,Z_1,Z_2,Z_3\rangle$, where Z_1, Z_2 and Z_3 may each be functions of the functions A, B, C, D, E, F, e, f and c; products of the fifth order coefficients§ $a_6^\dagger, b_6^\dagger\ldots\xi_6^\dagger, \zeta_6^\dagger$ and the factor $(-g/k)$ are denoted by $\tilde{a}, \tilde{b}, \ldots \tilde{\xi}, \tilde{\zeta}$; finally, we use $[u, v]$ to represent

$$-g\sqrt{\{\Phi(1+\epsilon\Phi)\}}\,(ugh'+vgg')/k^3.$$

‡ To derive these coefficients entails much laborious manipulation; the details of the mathematics are set out in Hawkes (1963).

§ The dagger indicates integration from z_o to z_c.

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$x_o r_o^4$: $\tilde{e} - \langle \Lambda;\ c^\dagger e^\dagger + 2(C^\dagger + D^\dagger)\, E^\dagger,\ e^\dagger e^* - 3E^\dagger E^* - 8A^*(C^\dagger + D^\dagger),\ 12A^*E^* \rangle$
$+ [e^\dagger e^* - 3E^\dagger E^*,\ c^\dagger e^\dagger + 2(C^\dagger + D^\dagger)\, E^\dagger]$,

$x_o r_a^4$: $\tilde{m} - \langle \Lambda;\ 12B^\dagger F^\dagger,\ f^\dagger f^* - 3F^\dagger F^* - 8B^\dagger(C^* + D^*),\ c^* f^* + 2(C^* + D^*)\, F^* \rangle$
$+ [f^\dagger f^* - F^\dagger F^* - 8B^\dagger C^*,\ 12B^\dagger F^\dagger]$,

$x_o r_o^2 r_a^2$: $\tilde{\lambda} - \langle \Lambda;\ c^\dagger f^\dagger + 2C^\dagger F^\dagger + 6D^\dagger F^\dagger + 4B^\dagger E^\dagger,\ e^\dagger f^* + e^* f^\dagger - E^\dagger F^* - 5E^* F^\dagger - 4C^\dagger D^*$
$- 4C^* D^\dagger - 16A^* B^\dagger - 4D^\dagger D^*,\ c^* e^* + 2C^* E^* + 4A^* F^* + 6D^* E^* \rangle$
$+ [e^\dagger f^* + e^* f^\dagger - E^\dagger F^* - 3E^* F^\dagger - 4C^* D^\dagger,\ c^\dagger f^\dagger + 2C^\dagger F^\dagger + 4B^\dagger E^\dagger + 6D^\dagger F^\dagger]$,

$x_o(\mathbf{r}_o . \mathbf{r}_a)^2$: $3\tilde{\jmath} - \langle \Lambda;\ 12C^\dagger F^\dagger,\ 3(c^\dagger c^* - 4C^\dagger C^* - 4E^* F^\dagger),\ 12C^* E^* \rangle$
$+ [4(c^\dagger c^* - 2C^\dagger C^* - E^* F^\dagger),\ 4(3C^\dagger F^\dagger - c^\dagger f^\dagger)]$,

$x_o(\mathbf{r}_o \times \mathbf{r}_a)^2$‡: $\tilde{v} - \langle \Lambda;\ 4c^\dagger f^\dagger,\ -3c^\dagger c^* - 4e^* f^\dagger,\ 4c^* e^* \rangle$,

$x_o(\mathbf{r}_o . \mathbf{r}_a)\,(\mathbf{r}_o \times \mathbf{r}_a)$: $2\tilde{u} - \langle \Lambda;\ 8(C^\dagger f^\dagger + c^\dagger F^\dagger),\ -8(E^* f^\dagger + e^* F^\dagger + C^* c^\dagger + c^* C^\dagger),\ 8(C^* e^* + c^* E^*) \rangle$
$+ [-4(c^\dagger C^* + E^* f^\dagger),\ 4(C^\dagger f^\dagger + c^\dagger F^\dagger)]$,

$x_o r_o^2(\mathbf{r}_o . \mathbf{r}_a)$: $2\tilde{f} - \langle \Lambda;\ c^{\dagger 2} + 4C^{\dagger 2} + 4E^\dagger F^\dagger + 8C^\dagger D^\dagger,$
$2c^\dagger e^* + 2c^* e^\dagger - 12C^\dagger E^* - 4C^* E^\dagger - 16A^* F^\dagger - 8D^\dagger E^*,\ 16A^* C^* + 8E^{*2} \rangle$
$+ [c^\dagger e^* - 6C^\dagger E^* + 4c^* e^\dagger - 4C^* E^\dagger - 4D^\dagger E^*,\ c^{\dagger 2} + 4C^{\dagger 2} - 4e^\dagger f^\dagger + 4E^\dagger F^\dagger + 8C^\dagger D^\dagger]$,

$x_o r_a^2(\mathbf{r}_o . \mathbf{r}_a)$: $2\tilde{n} - \langle \Lambda;\ 16B^\dagger C^\dagger + 8F^{\dagger 2},\ 2c^\dagger f^* + 2c^* f^\dagger - 4C^\dagger F^* - 12C^* F^\dagger - 16B^\dagger E^*$
$- 8D^* F^\dagger,\ c^{*2} + 4C^{*2} + 8C^* D^* + 4E^* F^* \rangle$
$+ [c^\dagger f^* - 2C^\dagger F^* + 4c^* f^\dagger - 8C^* F^\dagger - 8B^\dagger E^*,\ 16B^\dagger C^\dagger - 4f^{\dagger 2} + 8F^{\dagger 2}]$,

$x_o r_o^2(\mathbf{r}_o \times \mathbf{r}_a)$: $\tilde{\xi} - \langle \Lambda;\ 2(C^\dagger c^\dagger + 2c^\dagger D^\dagger + E^\dagger f^\dagger + e^\dagger F^\dagger),\ -6C^\dagger e^* - 2C^* e^\dagger - 2c^* E^\dagger - 4c^\dagger E^*$
$- 8A^* f^\dagger - 2D^\dagger e^*,\ 8(E^* e^* + A^* c^*) \rangle + [-3c^\dagger E^*,\ 2c^\dagger(C^\dagger + D^\dagger)]$,

$x_o r_a^2(\mathbf{r}_o \times \mathbf{r}_a)$: $\tilde{\zeta} - \langle \Lambda;\ 8(F^\dagger f^\dagger + B^\dagger c^\dagger),\ -6C^* f^\dagger - 2C^\dagger f^* - 4c^* F^\dagger - 2c^\dagger F^* - 8B^\dagger e^* - 4D^* f^\dagger,$
$2(C^* c^* + 2c^* D^* + e^* F^* + E^* f^*) \rangle + [-(c^\dagger F^* + 4C^* f^\dagger),\ 4(F^\dagger f^\dagger + B^\dagger c^\dagger)]$,

$-y_o r_o^4$: $\tilde{p} - \langle \Lambda;\ c^\dagger E^\dagger + 2D^\dagger e^\dagger,\ -e^\dagger E^* - 3e^* E^\dagger - 4A^* c^\dagger,\ 12A^* e^* \rangle$
$+ [-e^\dagger E^* - 3e^* E^\dagger,\ c^\dagger E^\dagger + 2e^\dagger D^\dagger]$,

$-y_o r_a^4$: $\tilde{s} - \langle \Lambda;\ 12B^\dagger f^\dagger,\ -F^\dagger f^* - 3F^* f^\dagger - 4B^\dagger c^*,\ c^* F^* + 2D^* f^* \rangle$
$+ [-3(F^\dagger f^* + F^* f^\dagger) - 8B^\dagger c^*,\ 36B^\dagger f^\dagger]$,

$-y_o r_o^2 r_a^2$: $\tilde{\mu} - \langle \Lambda;\ F^\dagger c^\dagger + 6D^\dagger F^\dagger + 4B^\dagger e^\dagger,\ -3(e^* F^\dagger + E^* f^\dagger) - 2(c^\dagger D^* + c^* D^\dagger)$
$- E^\dagger f^* - e^\dagger F^*,\ 4A^* f^* + c^* E^* + 6D^* e^* \rangle$
$+ [-(3e^* F^\dagger + E^* f^\dagger + 3E^\dagger f^* + 3e^\dagger F^* + 4c^* D^\dagger),\ 14D^\dagger f^\dagger + 12B^\dagger e^\dagger + c^\dagger F^\dagger]$,

$-y_o(\mathbf{r}_o . \mathbf{r}_a)^2$: $\tilde{u} - \langle \Lambda;\ 4(c^\dagger F^\dagger + C^\dagger f^\dagger),\ -4(E^* f^\dagger + e^* F^\dagger + c^* C^\dagger + C^* c^\dagger),\ 4(c^* E^* + C^* e^*) \rangle$
$+ [-4(2c^* C^\dagger + c^\dagger C^* + e^* F^\dagger),\ 8(C^\dagger f^\dagger + c^\dagger F^\dagger)]$,

$-y_o(\mathbf{r}_o \times \mathbf{r}_a)^2$: $3\tilde{r}$,

$-y_o(\mathbf{r}_o . \mathbf{r}_a)\,(\mathbf{r}_o \times \mathbf{r}_a)$: $2\tilde{v} - \langle \Lambda;\ 8c^\dagger f^\dagger,\ -6c^\dagger c^* - 8e^* f^\dagger,\ 8e^* c^* \rangle + [4(c^\dagger c^* + e^* f^\dagger),\ 8c^\dagger f^\dagger]$,

‡ This term may, of course, be incorporated into $x_o r_o^2 r_a^2$ and $x_o(\mathbf{r}_o . \mathbf{r}_a)^2$, by using the identity

$$(\mathbf{r}_o \times \mathbf{r}_a)^2 = r_o^2 r_a^2 - (\mathbf{r}_o . \mathbf{r}_a)^2.$$

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$-y_o r_o^2(\mathbf{r}_0\cdot\mathbf{r}_a)$: $\tilde{\xi}-\langle\Lambda;\ 2C^\dagger c^\dagger+4c^\dagger D^\dagger+2f^\dagger E^\dagger+2e^\dagger F^\dagger, -4c^\dagger E^*-6C^\dagger e^*-8A^*f^\dagger$
$-2c^*E^\dagger-2C^*e^\dagger-4D^\dagger e^*, 8(e^*E^*+A^*c^*)\rangle$
$+[-(6e^*C^\dagger+E^*c^\dagger+4c^*E^\dagger+4C^*e^\dagger+4D^\dagger e^*), 2c^\dagger C^\dagger+6c^\dagger D^\dagger+4E^\dagger f^\dagger+4e^\dagger f^\dagger]$,

$-y_o r_a^2(\mathbf{r}_o\cdot\mathbf{r}_a)$: $\tilde{\zeta}-\langle\Lambda;\ 8(F^\dagger f^\dagger+B^\dagger c^\dagger), -4c^*F^\dagger-6C^*f^\dagger-2c^\dagger F^*-2C^\dagger f^*-4D^*f^\dagger-8B^\dagger e^*,$
$2c^*C^*+4c^*D^*+2e^*F^*+2E^*f^*\rangle$
$+[-(6C^\dagger f^*+3c^\dagger F^*+8c^*F^\dagger+4C^*f^\dagger+8B^\dagger e^*), 20(F^\dagger f^\dagger+B^\dagger c^\dagger)]$,

$-y_o r_o^2(\mathbf{r}_o\times\mathbf{r}_a)$: $2\tilde{q}-\langle\Lambda;\ c^{\dagger 2}+4e^\dagger f^\dagger, -6c^\dagger e^*-2c^*e^\dagger, 8e^{*2}\rangle+[-3c^\dagger e^*, c^{\dagger 2}]$,

$-y_o r_a^2(\mathbf{r}_o\times\mathbf{r}_a)$: $2\tilde{t}-\langle\Lambda;\ 8f^{\dagger 2}, -6c^*f^\dagger-2c^\dagger f^*, c^{*2}+4e^*f^*\rangle+[-(3c^\dagger f^*+4c^*f^\dagger), 12f^{\dagger 2}]$,

$x_a r_o^4$: $2\tilde{b}-\langle\Lambda;\ 4D^{\dagger 2}+2e^\dagger f^\dagger+2E^\dagger F^\dagger, -4D^\dagger E^*-4D^*E^\dagger-8A^*F^\dagger, E^{*2}+e^{*2}+16A^*D^*\rangle$
$+[-4D^*E^\dagger-2D^\dagger E^*-2c^*e^\dagger, 2E^\dagger F^\dagger+6e^\dagger f^\dagger+4D^{\dagger 2}]$,

$x_a r_a^4$: $6\tilde{d}-\langle\Lambda;\ 48B^{\dagger 2}, -24B^\dagger F^*, 3(F^{*2}+f^{*2})\rangle+[-12B^\dagger F^*, 48B^{\dagger 2}]$,

$x_a r_o^2 r_a^2$: $4\tilde{c}-\langle\Lambda;\ 2(F^{\dagger 2}+f^{\dagger 2}+16B^\dagger D^\dagger), -8(D^*F^\dagger+D^\dagger F^*+2B^\dagger E^*), 4(2D^{*2}+E^*F^*+e^*f^*)\rangle$
$+[-4D^*F^\dagger-6D^\dagger F^*-4B^\dagger E^*-2c^*f^\dagger, 2F^{\dagger 2}+6f^{\dagger 2}+32B^\dagger D^\dagger]$,

$x_a(\mathbf{r}_o\cdot\mathbf{r}_a)^2$: $2\tilde{n}-\langle\Lambda;\ 8(F^{\dagger 2}+2B^\dagger C^\dagger), -12C^*F^\dagger-4C^\dagger F^*-8D^*F^\dagger-16B^\dagger E^*$
$+2c^\dagger f^*+2c^*f^\dagger, c^{*2}+4C^{*2}+8C^*D^*+4E^*F^*\rangle$
$+[-4C^\dagger F^*+2c^\dagger f^*-4C^*F^\dagger-8D^*F^\dagger, 8(F^{\dagger 2}+2B^\dagger C^\dagger)]$,

$x_a(\mathbf{r}_0\times\mathbf{r}_a)^2$: $2\tilde{t}-\langle\Lambda;\ 8f^{\dagger 2}, -6c^*f^\dagger-2c^\dagger f^*, c^{*2}+4e^*f^*\rangle$,

$x_a(\mathbf{r}_o\cdot\mathbf{r}_a)(\mathbf{r}_o\times\mathbf{r}_a)$: $2\tilde{\zeta}-\langle\Lambda;16(f^\dagger F^\dagger+B^\dagger c^\dagger), -4(3C^*f^\dagger+2c^*F^\dagger+c^\dagger F^*+C^\dagger f^*+2D^*f^\dagger+4B^\dagger e^*),$
$4(c^*C^*+2c^*D^*+e^*F^*+E^*f^*)\rangle+[-2c^\dagger F^*-4C^*f^\dagger-8D^*f^\dagger, 8(F^\dagger f^\dagger+B^\dagger c^\dagger)]$,

$x_a r_o^2(\mathbf{r}_o\cdot\mathbf{r}_a)$: $2\tilde{\lambda}-\langle\Lambda;\ 4C^\dagger F^\dagger+2c^\dagger f^\dagger+12D^\dagger F^\dagger+8B^\dagger E^\dagger,$
$-2(4C^\dagger D^*+4C^*D^\dagger+5E^*F^\dagger+E^\dagger F^*+16A^*B^\dagger+2D^\dagger D^*+e^\dagger f^*+e^*f^\dagger),$
$2(2C^*E^*+c^*e^*+6D^*E^*+4A^*F^*)\rangle$
$+[-2(4C^\dagger D^*+2C^*D^\dagger+E^*F^\dagger+E^\dagger F^*-e^\dagger f^*+c^\dagger c^*+4D^\dagger D^*),$
$4C^\dagger F^\dagger+6c^\dagger f^\dagger+12D^\dagger F^\dagger+8B^\dagger E^\dagger]$,

$x_a r_a^2(\mathbf{r}_o\cdot\mathbf{r}_a)$: $4\tilde{m}-\langle\Lambda;\ 48B^\dagger F^\dagger, -4(3F^\dagger F^*+f^\dagger f^*+8B^\dagger C^*+8B^\dagger D^*),$
$4(2C^*F^*+c^*f^*+2D^*F^*)\rangle+[2f^\dagger f^*-8F^\dagger F^*-8B^\dagger C^*-16B^\dagger D^*, 48B^\dagger F^\dagger]$,

$x_a r_o^2(\mathbf{r}_o\times\mathbf{r}_a)$: $2\tilde{\mu}-\langle\Lambda;\ 2c^\dagger F^\dagger-8e^\dagger B^\dagger+12D^\dagger F^\dagger,$
$-2(2c^\dagger D^*+2c^*D^\dagger+3e^*F^\dagger+3E^*f^\dagger+E^\dagger f^*+e^\dagger F^*),$
$12D^*e^*+2c^*E^*+8A^*f^*\rangle+[-4c^\dagger D^*-2E^*f^\dagger, 2c^\dagger F^\dagger+4D^\dagger f^\dagger]$,

$x_a r_a^2(\mathbf{r}_o\times\mathbf{r}_a)$: $4\tilde{s}-\langle\Lambda;\ 48B^\dagger f^\dagger, -4(3f^\dagger F^*+F^\dagger f^*+4B^\dagger c^*), 4c^*F^*+8D^*f^*\rangle$
$+[-6f^\dagger F^*, 24B^\dagger f^\dagger]$,

$y_a r_o^4$: $[-2(C^*e^\dagger+c^*E^\dagger-D^\dagger e^*), 2(e^\dagger F^\dagger+E^\dagger f^\dagger+c^\dagger D^\dagger)]$,

$y_a r_a^4$: $[F^\dagger f^*, 0]$,

$y_a r_o^2 r_a^2$: $[-2c^*F^\dagger-2C^*f^\dagger+4B^\dagger e^*+E^\dagger f^*, 4(F^\dagger f^\dagger+B^\dagger c^\dagger)]$,

$y_a(\mathbf{r}_o.\mathbf{r}_a)^2$: $[-2(2C^\dagger f^*+c^\dagger F^*), 8(F^\dagger f^\dagger+B^\dagger c^\dagger)]$

$y_a(\mathbf{r}_0.\mathbf{r}_a)\,(\mathbf{r}_o\times\mathbf{r}_a)$: $[-2c^\dagger f^*, 8f^{\dagger 2}]$,

$y_a r_o^2(\mathbf{r}_o.\mathbf{r}_a)$: $[-2(2c^*C^\dagger+c^\dagger C^*-e^*F+E^\dagger f^*+e^\dagger F^*), 4(C^\dagger f^\dagger+c^\dagger F^\dagger+2D^\dagger f^\dagger+2B^\dagger e^\dagger)]$,

$y_a r_a^2(\mathbf{r}_o.\mathbf{r}_a)$: $[2(C^\dagger f^*-f^*F^\dagger-f^\dagger F^*), 24B^\dagger f^\dagger]$,

$y_a r_o^2(\mathbf{r}_o\times\mathbf{r}_a)$: $[2(e^*f^\dagger-c^*c^\dagger), 4c^\dagger f^\dagger]$,

$y_a r_a^2(\mathbf{r}_o\times\mathbf{r}_a)$: $[c^\dagger f^*, 0]$.

This is the most convenient form in which to quote the aberration coefficients. The number of functions has been reduced to the fewest that is possible without returning to the functions $\Phi(z)$ and $\Omega(z)$ themselves; apart from $a_6^\dagger, \ldots, \zeta_6^\dagger$, only the nine functions $A(z), \ldots, c(z)$ are involved, together with the functions $\Lambda(z)$ and $\Lambda'(z)$.‡

In a later paper, the corresponding coefficients for a rectilinear orthogonal system will be set out.

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Appendix. Comparison with Melkich's formulae

The first case examined by Melkich corresponds to the special case of a twist-free orthogonal system in which $P_1 = \Delta_1 = 0$. Melkich's coefficients $a, b, c, \ldots, k, l$ in his expression

$$\Delta x_b = ax_B^3+bx_By_B^2+cx_B^2x_o+dx_By_By_o+ex_oy_B^2+fx_Bx_o^2+gx_By_o^2+jx_oy_oy_B+kx_o^3+lx_oy_o^2$$

correspond to the present coefficients $(\alpha\beta\gamma\delta)_x$ as follows.

a: $(0030) = k_x^{-1}(h_x n^*-4g_x c^\dagger)$, $\quad f$: $(2010) = k_x^{-1}(3h_x l^*-2g_x j^\dagger)$,

b: $(0012) = k_x^{-1}(h_x \phi^*-2g_x f^\dagger)$, $\quad g$: $(0210) = k_x^{-1}(h_x \lambda^*-2g_x h^\dagger)$,

c: $(1020) = k_x^{-1}(2h_x j^*-3g_x n^\dagger)$, $\quad j$: $(1101) = k_x^{-1}(2h_x \gamma^*-2g_x \delta^\dagger)$,

d: $(0111) = k_x^{-1}(h_x \omega^*-2g_x \zeta^\dagger)$, $\quad k$: $(3000) = k_x^{-1}(4h_x a^*-g_x l^\dagger)$,

e: $(1002) = k_x^{-1}(2h_x g^*-g_x \phi^\dagger)$, $\quad l$: $(1200) = k_x^{-1}(2h_x e^*-g_x \lambda^\dagger)$.

The y-aberrations are quoted in the plane of a line focus, in which $h_y(z)$ vanishes, so that the coefficients $\alpha, \beta, \ldots, \kappa, \lambda$ in

$$\Delta y_b = \alpha y_B^3+\beta y_Bx_B^2+\gamma y_B^2y_o+\delta y_Bx_Bx_o+\epsilon y_ox_B^2+\zeta y_By_o^2+\theta y_Bx_o^2+\iota y_ox_ox_B+\kappa y_o^3+\lambda y_ox_o^2$$

‡ In practice, therefore, we first calculate $m_{60}, m_{42}\ldots m_\Omega$ from (B 4) and hence $m_{300}, m_{210}\ldots m_{111}$ from (B 5) and (B 7). The paraxial solutions and primary aberrations are presumed known, so that the quantities $\bar{a}_6, \bar{b}_6\ldots\bar{\zeta}_6$ (B 10) can now be calculated and integrated to give $\tilde{a}, \tilde{b}\ldots\tilde{\zeta}$ (after multiplication by $-g/k$). The functions Λ, Λ' (B 11c) and $\mathscr{L}, \mathscr{Q}\ldots\mathscr{K}$ (B 14) are composed of known quantities, so that integrating, $J^\dagger, K^\dagger\ldots K^*, M^*$ can be obtained (B 13a and b). Any desired aberration coefficient can thus be calculated numerically.

correspond to $(\alpha\beta\gamma\delta)_y$ thus

$$\begin{aligned} \alpha&: (0003) = -4(g_y/k_y)\, d^\dagger, & \zeta&: (0201) = -2(g_y/k_y)\, k^\dagger \\ \beta&: (0021) = -2(g_y/k_y) f^\dagger, & \theta&: (2001) = -2(g_y/k_y)\, g^\dagger \\ \gamma&: (0102) = -3(g_y/k_y)\, p^\dagger, & \iota&: (1110) = -(g_y/k_y)\, \omega^\dagger \\ \delta&: (1011) = -2(g_y/k_y)\, \phi^\dagger, & \kappa&: (0300) = -(g_y/k_y)\, m^\dagger, \\ \epsilon&: (0120) = -(g_y/k_y)\, \zeta^\dagger, & \lambda&: (2100) = -(g_y/k_y)\, \gamma^\dagger. \end{aligned}$$

Melkich does not use the solutions of the primordial (Gaussian) equations of motion which we have denoted h_x and h_y. His functions x_γ and y_γ are identical to the present functions g_x and g_y, but his functions x_α and y_α are defined by

$$x_\alpha(z_o) = y_\alpha(z_o) = 0,$$

$$\left(\frac{\mathrm{d}x_\alpha}{\mathrm{d}z}\right)_{z=z_o} = \left(\frac{\mathrm{d}y_\alpha}{\mathrm{d}z}\right)_{z=z_o} = 1$$

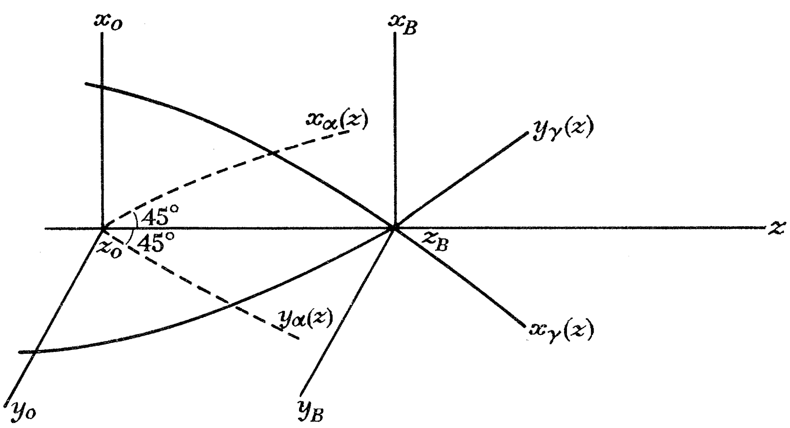

FIGURE 1.

(see figure 1). Melkich therefore has

$$\surd\Phi(x_\gamma x'_\alpha - x'_\gamma x_\alpha) = \surd\Phi_o$$

so that
$$x_\alpha(z_a) = -\sqrt{\frac{\Phi_o}{\Phi_a}}\frac{1}{x'_{\gamma a}}.$$

For g_x and h_x,
$$\surd\Phi(g_x h'_x - g'_x h_x) = \surd\Phi_o h'_{xo} = -g'_{xa}\surd\Phi_a$$

and hence
$$x_\alpha(z_a) = 1/h'_{xo} = x_{\alpha B}.$$

In conclusion, therefore,

$$x_\gamma(z) = g_x(z);$$

$$y_\gamma(z) = g_y(z);$$

$$x_\alpha(z) = \frac{1}{h'_{xo}} h_x(z), \quad h_x(z) = \frac{x_\alpha(z)}{x_{\alpha B}};$$

$$y_\alpha(z) = \frac{1}{h'_{yo}} h_y(z), \quad h_y(z) = \frac{y_\alpha(z)}{y_{\alpha B}}.$$

The potential expansions employed by Melkich also differ slightly from those employed in the present paper. The correspondence is as follows:

Melkich	Hawkes
$\Phi^*(z)$	$\Phi(z)$
$D(z)$	$D(z)$
$G(z)$	$-\frac{1}{4}\Phi^{(\mathrm{iv})}(z)+48D_1(z)$
$K(z)$	$-Q(z)$
$L(z)$	$48Q_1(z)$.

Let us examine the coefficient a, one of the aperture aberration coefficients

$$(\Delta x_b = a x_B^3+\ldots),$$

for which Melkich derives (p. 432) the following expression:

$$\begin{aligned} a = {} & \frac{x_{\alpha b}}{\sqrt{\Phi_o^*}\, x_{\alpha B}^3}\int_{z_B}^{z_b}\frac{x_\gamma}{\sqrt{\Phi^*}}\{\tfrac{1}{4}(\Phi''-D)\,x_\alpha^2 x_\alpha''+\tfrac{1}{8}(\Phi^{(3)}-D')\,x_\alpha^2 x_\alpha' \\ & -\tfrac{1}{4}(\Phi''-D-6\eta K\sqrt{\Phi^*})\,x_\alpha'^2 x_\alpha-\tfrac{1}{2}\Phi' x_\alpha'^3 \\ & +\tfrac{1}{24}[\Phi^{(4)}-D''+G-3(\eta K/\sqrt{\Phi^*})\,(\Phi''-D)-2\eta(K''+L)\sqrt{\Phi^*}]\,x_\alpha^3\}\,\mathrm{d}z \\ & -\frac{x_{\gamma b}}{\sqrt{\Phi_o^*}\, x_{\alpha B}^3}\int_{z_o}^{z_b}\frac{x_\alpha}{\sqrt{\Phi^*}}\{\tfrac{1}{4}(\Phi''-D)\,x_\alpha^2 x_\alpha''+\tfrac{1}{8}(\Phi^{(3)}-D')\,x_\alpha^2 x_\alpha' \\ & -\tfrac{1}{4}(\Phi''-D-6\eta K\sqrt{\Phi^*})\,x_\alpha'^2 x_\alpha-\tfrac{1}{2}\Phi' x_\alpha'^3 \\ & +\tfrac{1}{24}[\Phi^{(4)}-D''+G-3(\eta K/\sqrt{\Phi^*})\,(\Phi''-D)-2\eta(K''+L)\sqrt{\Phi^*}]\,x_\alpha^3\}\,\mathrm{d}z. \end{aligned}$$

We recall that x_α and x_γ are solutions of the 'Gaussian differential equation'

$$\Phi^* x''+\tfrac{1}{2}\Phi' x'+\tfrac{1}{4}(\Phi''-D-4\eta K\sqrt{\Phi^*})\,x = 0.$$

Consider first the second integral (with limits z_o and z_b);

$$\begin{aligned} -\frac{1}{2}\int_o^b\frac{\Phi'}{\sqrt{\Phi^*}}x_\alpha x_\alpha'^3\,\mathrm{d}z &= -\int_o^b x_\alpha x_\alpha'^3\,\mathrm{d}(\sqrt{\Phi^*})\,\mathrm{d}z \\ &= -[x_\alpha x_\alpha'^3\sqrt{\Phi^*}]_o^b+\int_o^b\sqrt{\Phi^*}\,(x_\alpha'^4+3x_\alpha x_\alpha'^2 x_\alpha'')\,\mathrm{d}z. \end{aligned}$$

But, from the Gaussian differential equation,

$$-\frac{1}{2}\frac{\Phi'}{\sqrt{\Phi^*}}x_\alpha x_\alpha'^3 = \frac{x_\alpha x_\alpha'^2}{\sqrt{\Phi^*}}\{\Phi^* x_\alpha''+\tfrac{1}{4}(\Phi''-D-4\eta K\sqrt{\Phi^*})x_\alpha\}.$$

Substituting for $\sqrt{\Phi^*}x_\alpha x_\alpha'^2 x_\alpha''$ from the first expression, we find

$$\begin{aligned} \frac{2}{3}\left(-\frac{1}{2}\int_o^b\frac{\Phi'}{\sqrt{\Phi^*}}x_\alpha x_\alpha'^3\,\mathrm{d}z\right) = {} & \frac{1}{4}\int_o^b\frac{\Phi''-D}{\sqrt{\Phi^*}}x_\alpha^2 x_\alpha'^2\,dz \\ & -\eta\int_o^b K x_\alpha^2 x_\alpha'^2\,\mathrm{d}z+\tfrac{1}{3}[x_\alpha x_\alpha'^3\sqrt{\Phi^*}]_o^b-\frac{1}{3}\int_o^b\sqrt{\Phi^*}x_\alpha'^4\,\mathrm{d}z, \end{aligned}$$

or

$$\begin{aligned} -\frac{1}{2}\int_o^b\left(\frac{\Phi'}{\sqrt{\Phi^*}}x_\alpha x_\alpha'^3-3\eta K x_\alpha'^2 x_\alpha^2\right)\mathrm{d}z = {} & \frac{3}{8}\int_o^b\frac{\Phi''-D}{\sqrt{\Phi^*}}x_\alpha^2 x_\alpha'^2\,\mathrm{d}z \\ & -\frac{1}{2}\int_o^b\sqrt{\Phi^*}x_\alpha'^4\,\mathrm{d}z+\tfrac{1}{2}\sqrt{\Phi_b^*}\,x_{\alpha b}x_{\alpha b}'^3 \qquad \text{(C 1)} \end{aligned}$$

in which we have used the fact that $x_\alpha(z_o) = 0$.

We next consider the mixed magnetic and electric term, $-\frac{1}{8}(\eta K/\surd\Phi^*)\,(\Phi''-D)\,x_\alpha^4$, and use the Gaussian equation to obtain

$$-\frac{1}{8}\frac{\eta K}{\Phi^*}(\Phi''-D)\,x_\alpha^4 = -\frac{1}{8}\,\frac{\Phi''-D}{\Phi^{*\frac{3}{2}}}\{\Phi^* x_\alpha''+\tfrac{1}{2}\Phi' x_\alpha'+\tfrac{1}{4}(\Phi''-D)\,x_\alpha\}\,x_\alpha^3. \tag{C 2}$$

Collecting up the terms in $x_\alpha^3\,x_\alpha''$, we find

$$\frac{1}{8}\int_o^b \frac{\Phi''-D}{\surd\Phi^*}\,x_\alpha^3\,x_\alpha''\,\mathrm{d}z$$

which we convert into

$$\frac{1}{8}\left[\frac{\Phi''-D}{\surd\Phi^*}\,x_\alpha^3\,x_\alpha'\right]_o^b-\frac{1}{8}\int_o^b x_\alpha'\left(3\,\frac{\Phi''-D}{\surd\Phi^*}\,x_\alpha^2\,x_\alpha'+\frac{\Phi^{(3)}-D'}{\surd\Phi^*}\,x_\alpha^3-\frac{\Phi''-D}{2\Phi^{*\frac{3}{2}}}\,\Phi' x_\alpha^3\right)\mathrm{d}z. \tag{C 3}$$

The aggregate of the terms in $x_\alpha^3\,x_\alpha'$ is zero.

With the aid of (C 1), (C 2) and (C 3), therefore, the second integrand of a can be written in the form

$$\int_o^b (Ax_\alpha^4+Bx_\alpha^2\,x_\alpha'^2+Cx_\alpha'^4)\,\mathrm{d}z+R$$

in which

$$A=\frac{1}{24}\,\frac{\Phi^{(4)}-D''+G}{\surd\Phi^*}-\frac{1}{32}\,\frac{(\Phi''-D)^2}{\Phi^{*\frac{3}{2}}}-\tfrac{1}{12}\eta(K''+L),\quad B=-\frac{1}{4}\,\frac{\Phi''-D}{\surd\Phi^*},\quad C=-\tfrac{1}{2}\surd\Phi^*,$$

$$R=\frac{1}{8}\,\frac{\Phi_b''-D_b}{\surd\Phi_b^*}\,x_{\alpha b}^3\,x_{\alpha b}'+\tfrac{1}{2}\surd\Phi_b^*\,x_{\alpha b}\,x_{\alpha b}'^3.$$

In the present notation, therefore,

$$-\frac{x_{\gamma b}}{\surd\Phi^*\,x_{\alpha B}^3}\int_{z_o}^{z_b}\frac{x_\alpha}{\surd\Phi^*}\{\ldots\}\,\mathrm{d}z$$

becomes

$$-\frac{g_{xc}}{k_x}\int_{z_o}^{z_c}(Ah_x^4+Bh_x^2\,h_x'^2+Ch_x'^4)\,\mathrm{d}z-R_1$$

in which

$$A=\frac{1}{\surd\Phi}\,(\tfrac{1}{32}\Phi^{(4)}-\tfrac{1}{24}D''+2D_1)-\frac{1}{32}\,\frac{(\Phi''-D)^2}{\Phi^{\frac{3}{2}}}+\tfrac{1}{12}\eta(Q''-48Q_1),\quad B=-\frac{1}{4}\,\frac{\Phi''-D}{\surd\Phi},$$

$$C=-\tfrac{1}{2}\surd\Phi,\quad R_1=\frac{1}{8}\,\frac{\Phi_c''-D_c}{k_x\surd\Phi_c}\,g_{xc}\,h_{xc}^3\,h_{xc}'+\frac{1}{2}\,\frac{\surd\Phi_c}{k_x}\,g_{xc}\,h_{xc}\,h_{xc}'^3.$$

Substituting for $F, G, \ldots, N$ in the expression for $\bar{c}$ obtained in part A, and neglecting relativistic effects, we find

$$4\bar{c}=A'h_x^4+B'h_x^2\,h_x'^2+C'h_x'^4$$

in which

$$A'=\frac{1}{\surd\Phi}\,(\tfrac{1}{32}\Phi^{(4)}-\tfrac{1}{24}D''+2D_1)-\frac{1}{32}\,\frac{(\Phi''-D)^2}{\Phi^{\frac{3}{2}}}+\tfrac{1}{12}\eta(Q''-48Q_1)$$

$$B'=-\frac{1}{4}\,\frac{\Phi''-D}{\surd\Phi},\quad C'=-\tfrac{1}{2}\surd\Phi.$$

A and A', B and B', C and C' are thus respectively identical.

We now transform the first integral in Melkich's expression, which runs from z_B to z_b. The sequence of the steps is identical to the one we have just followed to convert the second integral.

$$-\frac{1}{2}\int \frac{\Phi'}{\sqrt{\Phi^*}} x_\gamma x_\alpha'^3 \,dz = \int \sqrt{\Phi^*}\, x_\gamma x_\alpha'^2 x_\alpha'' \,dz + \frac{1}{4}\int \frac{\Phi''-D}{\sqrt{\Phi^*}} x_\gamma x_\alpha x_\alpha'^2 \,dz - \eta \int K x_\gamma x_\alpha x_\alpha'^2 \,dz$$

and $$-\frac{1}{2}\int_B^b \frac{\Phi'}{\sqrt{\Phi^*}} x_\gamma x_\alpha'^3 \,dz = -[\sqrt{\Phi^*}\, x_\gamma x_\alpha'^3]_B^b + \int_B^b \sqrt{\Phi^*}\, x_\gamma' x_\alpha'^3 \,dz + 3\int_B^b \sqrt{\Phi^*}\, x_\gamma x_\alpha'^2 x_\alpha'' \,dz$$

give $$-\frac{1}{2}\int_B^b \frac{\Phi'}{\sqrt{\Phi^*}} x_\gamma x_\alpha'^3 \,dz + \tfrac{3}{2}\eta \int_B^b K x_\gamma x_\alpha x_\alpha'^2 \,dz$$

$$= \frac{3}{8}\int_B^b \frac{\Phi''-D}{\sqrt{\Phi^*}} x_\gamma x_\alpha x_\alpha'^2 \,dz - \frac{1}{2}\int_B^b \sqrt{\Phi^*}\, x_\gamma' x_\alpha'^3 \,dz + \tfrac{1}{2}\sqrt{\Phi_b^*}\, x_{\gamma b} x_{\alpha b}'^3 \qquad \text{(C 4)}$$

$$-\tfrac{1}{8}\eta \int \frac{K}{\Phi^*}(\Phi''-D)\, x_\gamma x_\alpha^3 \,dz = -\frac{1}{8}\int \frac{\Phi''-D}{\Phi^{*\frac{3}{2}}}[\Phi^* x_\alpha'' + \tfrac{1}{2}\Phi' x_\alpha' + \tfrac{1}{4}(\Phi''-D)\, x_\alpha]\, x_\gamma x_\alpha^2 \,dz. \qquad \text{(C 5)}$$

The terms in $x_\gamma x_\alpha^2 x_\alpha''$,

$$\int_B^b \frac{1}{8}\frac{\Phi''-D}{\sqrt{\Phi^*}} x_\gamma x_\alpha^2 x_\alpha'' \,dz$$

are written

$$\frac{1}{8}\left(\frac{\Phi''-D}{\sqrt{\Phi^*}}\right)_b x_{\gamma b} x_{\alpha b}^2 x_{\alpha b}' - \frac{1}{8}\int_B^b x_\alpha' \left(\frac{\Phi^{(3)}-D'}{\sqrt{\Phi^*}} x_\gamma x_\alpha^2 - \frac{\Phi''-D}{2\Phi^{*\frac{3}{2}}}\Phi' x_\gamma x_\alpha^2 + \frac{\Phi''-D}{\sqrt{\Phi^*}} x_\gamma' x_\alpha^2 + 2\frac{\Phi''-D}{\sqrt{\Phi^*}} x_\gamma x_\alpha x_\alpha'\right) dz \qquad \text{(C 6)}$$

and those in $x_\gamma x_\alpha^2 x_\alpha'$ are hence equal to zero.

The complete first integrand in a can thus be cast into the form

$$\int_B^b (U x_\gamma x_\alpha^3 + V x_\alpha x_\alpha'(x_\gamma x_\alpha' + x_\gamma' x_\alpha) + W x_\gamma' x_\alpha'^3)\,dz + R'$$

in which

$$U = \frac{1}{24}\frac{\Phi^{(4)}-D''+G}{\sqrt{\Phi^*}} - \frac{1}{32}\frac{(\Phi''-D)^2}{\Phi^{*\frac{3}{2}}} - \tfrac{1}{12}\eta(K''+L), \qquad V = -\frac{1}{8}\frac{\Phi''-D}{\sqrt{\Phi^*}},$$

$$W = -\tfrac{1}{2}\sqrt{\Phi^*}, \qquad R' = \frac{1}{8}\left(\frac{\Phi''-D}{\sqrt{\Phi^*}}\right)_b x_{\gamma b} x_{\alpha b}^2 x_{\alpha b}' + \tfrac{1}{2}\sqrt{\Phi_b^*}\, x_{\gamma b} x_{\alpha b}'^3$$

so that $U = A = A'$, $V = \tfrac{1}{2}B = \tfrac{1}{2}B'$ and $W = C = C'$.

The first integral

$$\frac{x_{\alpha b}}{\sqrt{\Phi_o^*}\, x_{\alpha B}^3}\int_{z_B}^{z_b} \frac{x_\gamma}{\sqrt{\Phi^*}}\{\ldots\}\,dz$$

is thus equivalent to

$$\frac{h_{xc}}{k_x}\int_{z_a}^{z_c} (A g_x h_x^3 + \tfrac{1}{2}B(g_x h_x)' h_x h_x' + C g_x' h_x'^3)\,dz + R'$$

in which $$R' = \frac{1}{8}\frac{\Phi_c''-D_c}{k_x\sqrt{\Phi_c}} g_{xc} h_{xc}^3 h_{xc}' + \tfrac{1}{2}\sqrt{\Phi_c}\, g_{xc} h_{xc} h_{xc}'^3.$$

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Combining the first and second integrals, the terms in R and R' cancel out, and since we found in part A that the expression for $\bar{n}$ is indeed equal to

$$Ag_x h_x^3 + \tfrac{1}{2}B(g_x h_x)' h_x h_x' + Cg_x' h_x'^3,$$

Melkich's expression for a and the formula derived above for the corresponding quantity are identical.

The expression given by Melkich for the aperture aberration coefficient in the y-direction, $y_b = \alpha y_B^3$ (equation (20), p. 428), is the counterpart of only the second integral in the x direction, or $-4(g_{yc}/k_y)\,d^\dagger$ in the present notation; since the symmetry of the coefficients is so high, he finds it convenient to quote the aberrations at a line focus in this, the $y\,z$ plane, and to set out the formulae for a general plane in only the $x\,z$ plane. The term in R can no longer cancel out with the term in R', therefore, but this is of no importance, since x_α vanishes in a line focus plane and hence $R = 0$.

In the remainder of this appendix, we shall demonstrate briefly the equivalence of Melkich's expressions and the present formulae, for one member of each family of aberrations: one coma, one astigmatism and one distortion.

A coma term: $\Delta x_b = dx_B y_B y_o$

Melkich finds

$$\begin{aligned}
d = \frac{x_{\alpha b}}{\sqrt{\Phi_o^*}\, x_{\alpha B} y_{\alpha B}} \int_{z_B}^{z_b} \frac{x_\gamma}{\sqrt{\Phi^*}} \{&\tfrac{1}{2}(\Phi''+D)\, x_\alpha'' y_\alpha y_\gamma - 2\Phi^* x_\alpha'' y_\alpha' y_\gamma' \\
&+\tfrac{1}{4}(\Phi^{(3)}+D')\, x_\alpha' y_\alpha y_\gamma + \tfrac{1}{4}(\Phi''+D)\,(x_\alpha' y_\alpha y_\gamma' + x_\alpha' y_\gamma y_\alpha') \\
&-\Phi' x_\alpha' y_\alpha' y_\gamma' + \Phi^*(x_\alpha' y_\alpha'' y_\gamma' + x_\alpha' y_\gamma'' y_\alpha') \\
&-(\Phi''-D-3\eta K\sqrt{\Phi^*})\, x_\alpha y_\alpha' y_\gamma' - \tfrac{1}{4}[(\eta K/\sqrt{\Phi^*})\,(\Phi''+D) \\
&+2\eta(K''-L)\sqrt{\Phi^*}+G]\, x_\alpha y_\alpha y_\gamma - \eta K'\sqrt{\Phi^*}(x_\alpha y_\alpha y_\gamma' + x_\alpha y_\alpha' y_\gamma)\}\,\mathrm{d}z \\
-\frac{x_{\gamma b}}{\sqrt{\Phi_o^*}\, x_{\alpha B} y_{\alpha B}} \int_{z_o}^{z_b} \frac{x_\alpha}{\sqrt{\Phi^*}} \{&\tfrac{1}{2}(\Phi''+D)\, x_\alpha'' y_\alpha y_\gamma - 2\Phi^* x_\alpha'' y_\alpha' y_\gamma' \\
&+\tfrac{1}{4}(\Phi^{(3)}+D')\, x_\alpha' y_\alpha y_\gamma + \tfrac{1}{4}(\Phi''+D)\,(x_\alpha' y_\alpha y_\gamma' + x_\alpha' y_\gamma y_\alpha') \\
&-\Phi' x_\alpha' y_\alpha' y_\gamma' + \Phi^*(x_\alpha' y_\alpha'' y_\gamma' + x_\alpha' y_\gamma'' y_\alpha') \\
&-(\Phi''-D-3\eta K\sqrt{\Phi^*})\, x_\alpha y_\alpha' y_\gamma' - \tfrac{1}{4}[(\eta K/\sqrt{\Phi^*})\,(\Phi''+D) \\
&+2\eta(K''-L)\sqrt{\Phi^*}+G]\, x_\alpha y_\alpha y_\gamma - \eta K'\sqrt{\Phi^*}(x_\alpha y_\alpha y_\gamma' + x_\alpha y_\alpha' y_\gamma)\}\,\mathrm{d}z.
\end{aligned}$$

Examine first the integral from z_B to z_b, which we expect to be equivalent to ω^*.

$$-\tfrac{1}{4}\eta K \frac{\Phi''+D}{\Phi^*} x_\gamma x_\alpha y_\gamma y_\alpha = -\frac{1}{4}\frac{\Phi''+D}{\Phi^{*\frac{3}{2}}} x_\gamma y_\gamma y_\alpha [\Phi^* x_\alpha'' + \tfrac{1}{2}\Phi' x_\alpha' + \tfrac{1}{4}(\Phi''-D)\, x_\alpha], \tag{C 7}$$

$$\begin{aligned}
\frac{1}{4}\int \frac{\Phi''+D}{\sqrt{\Phi^*}} x_\gamma x_\alpha'' y_\gamma y_\alpha\,\mathrm{d}z = {} & \left[\frac{1}{4}\frac{\Phi''+D}{\sqrt{\Phi^*}} x_\gamma x_\alpha' y_\gamma y_\alpha\right]_{z_B}^{z_b} \\
& -\frac{1}{4}\int\left[\frac{\Phi^{(3)}+D'}{\sqrt{\Phi^*}} x_\gamma x_\alpha' y_\gamma y_\alpha - \frac{\Phi''+D}{2\sqrt{\Phi^*}}\Phi' x_\gamma x_\alpha' y_\gamma y_\alpha\right. \\
& \left. + \frac{\Phi''+D}{\sqrt{\Phi^*}} x_\gamma' x_\alpha' y_\gamma y_\alpha + \frac{\Phi''+D}{\sqrt{\Phi^*}} x_\gamma x_\alpha' (y_\gamma y_\alpha)'\right]\mathrm{d}z,
\end{aligned} \tag{C 8}$$

$$\begin{aligned}
-\int_B^b \frac{\Phi'}{\sqrt{\Phi^*}} x_\gamma x_\alpha' y_\gamma' y_\alpha'\,\mathrm{d}z = {} & -[2\sqrt{\Phi^*}\, x_\gamma x_\alpha' y_\gamma' y_\alpha']_B^b \\
& +2\int_B^b \sqrt{\Phi^*}[x_\gamma' x_\alpha' y_\gamma' y_\alpha' + x_\gamma x_\alpha'' y_\gamma' y_\alpha' + x_\gamma x_\alpha'(y_\gamma'' y_\alpha' + y_\gamma' y_\alpha'')]\,\mathrm{d}z,
\end{aligned}$$

or alternatively,

$$-\int_B^b \frac{\Phi'}{\sqrt{\Phi^*}} x_\gamma x'_\alpha y'_\gamma y'_\alpha \,\mathrm{d}z = 2\int x_\gamma y'_\gamma y'_\alpha \left(\sqrt{\Phi^*}\, x''_\alpha + \frac{1}{4}\frac{\Phi''-D}{\sqrt{\Phi^*}} x_\alpha - \eta K x_\alpha\right) \mathrm{d}z.$$

Multiplying the first of these expressions by $\frac{1}{2}$ and the second by $\frac{3}{2}$, and subtracting, we find

$$\begin{aligned} -\int_B^b \frac{\Phi'}{\sqrt{\Phi^*}} x_\gamma x'_\alpha y'_\gamma y'_\alpha \,\mathrm{d}z &= 2\int_B^b \sqrt{\Phi^*}\, x_\gamma x''_\alpha y'_\gamma y'_\alpha \,\mathrm{d}z \\ &+\frac{3}{4}\int_B^b \frac{\Phi''-D}{\sqrt{\Phi^*}} x_\gamma x_\alpha y'_\gamma y'_\alpha \,\mathrm{d}z - \int_B^b \sqrt{\Phi^*}\, x'_\gamma x'_\alpha y'_\gamma y'_\alpha \,\mathrm{d}z \\ &-\int_B^b \sqrt{\Phi^*}\, x_\gamma x'_\alpha (y''_\gamma y'_\alpha + y'_\gamma y''_\alpha) \mathrm{d}z - 3\eta \int_B^b K x_\gamma x_\alpha y'_\gamma y'_\alpha \,\mathrm{d}z \\ &+ [\sqrt{\Phi^*}\, x_\gamma x'_\alpha y'_\gamma y'_\alpha]_B^b. \end{aligned} \tag{C 9}$$

Finally,

$$\begin{aligned} -\eta\int_B^b K' x_\gamma x_\alpha (y_\gamma y_\alpha)' \,\mathrm{d}z &= -\tfrac{1}{2}\eta \int_B^b K' x_\gamma x_\alpha (y_\gamma y_\alpha)' \,\mathrm{d}z \\ &- [\tfrac{1}{2}\eta K' x_\gamma x_\alpha y_\gamma y_\alpha]_B^b + \int_B^b \tfrac{1}{2}\eta K' (x_\gamma x_\alpha)' y_\gamma y_\alpha \,\mathrm{d}z \\ &+ \int_B^b \tfrac{1}{2}\eta K'' x_\gamma x_\alpha y_\gamma y_\alpha \,\mathrm{d}z. \end{aligned}$$

Collecting up, we obtain

$$\begin{aligned} &\frac{x_{\alpha b}}{\sqrt{\Phi_o^*}\, x_{\alpha B}\, y_{\alpha B}} \int_{z_B}^{z_b} \frac{x_\gamma}{\sqrt{\Phi^*}} \{\ldots\} \mathrm{d}z \\ &= \frac{x_{\alpha b}}{\sqrt{\Phi_o^*} x_{\alpha B} y_{\alpha B}} \int_{z_B}^{z_b} [T x_\gamma x_\alpha y_\gamma y_\alpha + U x'_\gamma x'_\alpha y'_\gamma y'_\alpha \\ &\quad + V_+ (x_\gamma x_\alpha y'_\gamma y'_\alpha + x'_\gamma x'_\alpha y_\gamma y_\alpha) + V_- (x_\gamma x_\alpha y'_\gamma y'_\alpha - x'_\gamma x'_\alpha y_\gamma y_\alpha) \\ &\quad + W\{x_\gamma x_\alpha (y_\gamma y_\alpha)' - (x_\gamma x_\alpha)' y_\gamma y_\alpha\}]\, \mathrm{d}z + R_1 \\ &= \frac{h_{xc}}{k_x} \int_{z_a}^{z_c} [T g_x h_x g_y h_y + U g'_x h'_x g'_y h'_y \\ &\quad + V_+ (g_x h_x g'_y h'_y + g'_x h'_x g_y h_y) + V_- (g_x h_x g'_y h'_y - g'_x h'_x g_y h_y) \\ &\quad + W\{g_x h_x (g_y h_y)' - (g_x h_x)' g_y h_y\}]\, \mathrm{d}z + R_1 \end{aligned}$$

in which

$$\begin{aligned} T &= -\frac{G}{4\sqrt{\Phi^*}} - \frac{1}{16}\frac{(\Phi''-D)(\Phi''+D)}{\Phi^{*\frac{3}{2}}} + \tfrac{1}{2}\eta L \\ &= \frac{1}{16}\frac{\Phi^{(\mathrm{iv})}}{\sqrt{\Phi}} - 12\frac{D_1}{\sqrt{\Phi}} - \frac{(\Phi''-D)(\Phi''+D)}{16\Phi^{\frac{3}{2}}} + 24\eta Q_1, \end{aligned}$$

or $8(F-G)$ in the notation of part A.

$$U = -\sqrt{\Phi^*} = -\sqrt{\Phi} \quad \text{or} \quad 8N,$$

$$V_+ = -\frac{1}{4}\frac{\Phi''}{\sqrt{\Phi^*}} = -\frac{1}{4}\frac{\Phi''}{\sqrt{\Phi}} \quad \text{or} \quad 4K,$$

$$V_- = \frac{1}{4}\frac{D}{\sqrt{\Phi^*}} = \frac{1}{4}\frac{D}{\sqrt{\Phi}} \quad \text{or} \quad 4L,$$

$$W = -\tfrac{1}{2}\eta K' = \tfrac{1}{2}\eta Q' \quad \text{or} \quad 2R.$$

The 'remainder', R_1, is given by

$$R_1 = \frac{h_{xc}}{k_x}\left[\frac{1}{4}\frac{\Phi''+D}{\sqrt{\Phi}} g_x h_x' g_y h_y + \sqrt{\Phi} g_x h_x' g_y' h_y' + \tfrac{1}{2}\eta Q' g_x h_x g_y h_y\right]_{x_a}^{x_c}.$$

Precisely the same sequence of steps converts the second integral, from z_o to z_b, into an expression which proves to be identical to $2\zeta^\dagger$, together with a remainder which cancels out R_1.

An astigmatism term: $\Delta x_b = g x_B y_o^2$

Melkich derives the formula

$$\begin{aligned} g = \frac{x_{\alpha b}}{\sqrt{\Phi_o^*}\, x_{\alpha B}} \int_{z_B}^{z_b} \frac{x_\gamma}{\sqrt{\Phi^*}} \{ & \tfrac{1}{4}(\Phi''+D)\, x_\alpha'' y_\gamma^2 - \Phi^* x_\alpha'' y_\gamma'^2 \\ & + \tfrac{1}{8}(\Phi^{(3)}+D')\, x_\alpha' y_\gamma^2 + \tfrac{1}{4}(\Phi''+D)\, x_\alpha' y_\gamma y_\gamma' - \tfrac{1}{2}\Phi' x_\alpha' y_\gamma'^2 \\ & + \Phi^* x_\alpha' y_\gamma' y_\gamma'' - \tfrac{1}{2}(\Phi''-D-3\eta K\sqrt{\Phi^*})\, x_\alpha y_\gamma'^2 \\ & - \tfrac{1}{8}[(\eta K/\sqrt{\Phi^*})\,(\Phi''+D) + 2\eta(K''-L)\sqrt{\Phi^*} + G]\, x_\alpha y_\gamma^2 - \eta K'\sqrt{\Phi^*}\, x_\alpha y_\gamma y_\gamma' \}\, dz \\ - \frac{x_{\gamma b}}{\sqrt{\Phi_o^*}\, x_{\alpha B}} \int_{z_o}^{z_b} \frac{x_\alpha}{\sqrt{\Phi^*}} \{ & \tfrac{1}{4}(\Phi''+D)\, x_\alpha'' y_\gamma^2 - \Phi^* x_\alpha'' y_\gamma'^2 \\ & + \tfrac{1}{8}(\Phi^{(3)}+D')\, x_\alpha' y_\gamma^2 + \tfrac{1}{4}(\Phi''+D)\, x_\alpha' y_\gamma y_\gamma' - \tfrac{1}{2}\Phi' x_\alpha' y_\gamma'^2 \\ & + \Phi^* x_\alpha' y_\gamma' y_\gamma'' - \tfrac{1}{2}(\Phi''-D-3\eta K\sqrt{\Phi^*})\, x_\alpha y_\gamma'^2 \\ & - \tfrac{1}{8}[(\eta K/\sqrt{\Phi^*})\,(\Phi''+D) + 2\eta(K''-L)\sqrt{\Phi^*} + G]\, x_\alpha y_\gamma^2 - \eta K'\sqrt{\Phi^*}\, x_\alpha y_\gamma y_\gamma' \}\, dz. \end{aligned}$$

We find that

$$-\frac{1}{2}\int \frac{\Phi'}{\sqrt{\Phi^*}} x_\gamma x_\alpha' y_\gamma'^2\, dz = \int x_\gamma y_\gamma'^2 \left(\sqrt{\Phi^*}\, x_\alpha'' + \frac{1}{4}\frac{\Phi''-D}{\sqrt{\Phi^*}} x_\alpha - \eta K x_\alpha\right) dz$$

and

$$-\frac{1}{2}\int_B^b \frac{\Phi'}{\sqrt{\Phi^*}} x_\gamma x_\alpha' y_\gamma'^2\, dz = -[\sqrt{\Phi^*}\, x_\gamma x_\alpha' y_\gamma'^2]_B^b + \int_B^b \sqrt{\Phi^*}\,(x_\gamma' x_\alpha' y_\gamma'^2 + x_\gamma x_\alpha'' y_\gamma'^2 + 2 x_\gamma x_\alpha' y_\gamma' y_\gamma'')\, dz$$

give

$$\begin{aligned} -\frac{1}{2}\int_B^b \frac{\Phi'}{\sqrt{\Phi^*}} x_\gamma x_\alpha' y_\gamma'^2\, dz + \tfrac{3}{2}\eta \int_B^b K x_\gamma x_\alpha y_\gamma'^2\, dz - \int_B^b \sqrt{\Phi^*}\, x_\gamma x_\alpha'' y_\gamma'^2\, dz \\ + \int_B^b \sqrt{\Phi^*}\, x_\gamma x_\alpha' y_\gamma' y_\gamma''\, dz = -\frac{1}{2}\int_B^b \sqrt{\Phi^*}\, x_\gamma' x_\alpha' y_\gamma'^2\, dz \\ + \frac{3}{8}\int_B^b \frac{\Phi''-D}{\sqrt{\Phi^*}} x_\gamma x_\alpha y_\gamma'^2\, dz + \tfrac{1}{2}[\sqrt{\Phi^*}\, x_\gamma x_\alpha' y_\gamma'^2]_B^b, \end{aligned} \quad \text{(C 10)}$$

$$-\tfrac{1}{8}\eta \int \frac{K}{\sqrt{\Phi^*}}(\Phi''+D)\, x_\gamma x_\alpha y_\gamma^2\, dz = -\frac{1}{8}\int \frac{\Phi''+D}{\Phi^{*\frac{3}{2}}}[\Phi' x_\alpha'' + \tfrac{1}{2}\Phi' x_\alpha' + \tfrac{1}{4}(\Phi''-D)\, x_\alpha]\, x_\gamma y_\gamma^2\, dz, \quad \text{(C 11)}$$

$$\begin{aligned} -\eta \int_B^b K' x_\gamma x_\alpha y_\gamma y_\gamma'\, dz = -\tfrac{1}{4}\eta \int_B^b K' x_\gamma x_\alpha (y_\gamma^2)'\, dz \\ -\tfrac{1}{4}\eta\, [K' x_\gamma x_\alpha y_\gamma^2]_B^b + \tfrac{1}{4}\eta \int_B^b K'' x_\gamma x_\alpha y_\gamma^2\, dz + \tfrac{1}{4}\eta \int K' y_\gamma^2 (x_\gamma x_\alpha)'\, dz \end{aligned}$$

$$\begin{aligned} \frac{1}{8}\int_B^b \frac{\Phi''+D}{\sqrt{\Phi^*}} x_\gamma x_\alpha'' y_\gamma^2\, dz = \frac{1}{8}\left[\frac{\Phi''+D}{\sqrt{\Phi^*}} x_\gamma x_\alpha' y_\gamma^2\right]_B^b \\ - \int_B^b \left\{\frac{1}{8}\frac{\Phi'''+D'}{\sqrt{\Phi^*}} x_\gamma x_\alpha' y_\gamma^2 - \frac{1}{16}\frac{\Phi''+D}{\Phi^{*\frac{3}{2}}}\Phi' x_\gamma x_\alpha' y_\gamma^2 \right. \\ \left. + \frac{1}{8}\frac{\Phi''+D}{\sqrt{\Phi^*}} x_\gamma' x_\alpha' y_\gamma^2 + \frac{1}{8}\frac{\Phi''+D}{\sqrt{\Phi^*}} x_\gamma x_\alpha' (y_\gamma^2)'\right\} dz. \end{aligned} \quad \text{(C 12)}$$

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Collecting up, we find that

$$\frac{x_{\alpha B}}{\sqrt{\Phi_o^*}\,x_{\alpha B}}\int_{z_B}^{z_b}\frac{x_\gamma}{\sqrt{\Phi^*}}\{\ldots\}\,\mathrm{d}z$$

$$=\frac{x_{\alpha B}}{\sqrt{\Phi_o^*}\,x_{\alpha B}}\int_{z_B}^{z_b}[tx_\gamma x_\alpha y_\gamma^2+ux'_\gamma x'_\alpha y'^2_\gamma+v_+(x'_\gamma x'_\alpha y_\gamma^2+x_\gamma x_\alpha y'^2_\gamma)+v_-(x'_\gamma x'_\alpha y_\gamma^2-x_\gamma x_\alpha y'^2_\gamma)$$
$$+w\{x_\gamma x_\alpha(y_\gamma^2)'+(x_\gamma x_\alpha)'\,y_\gamma^2\}]\,\mathrm{d}z+R_2$$

$$=\frac{h_{xc}}{k_x}\int_{z_a}^{z_c}[tg_x h_x g_y^2+ug'_x h'_x g'^2_y+v_+(g'_x h'_x g_y^2+g_x h_x g'^2_y)+v_-(g'_x h'_x g_y^2-g_x h_x g'^2_y)$$
$$+w\{g_x h_x(g_y^2)'-(g_x h_x)'\,g_y^2\}]\,\mathrm{d}z+R_2$$

in which
$$t=-\frac{G}{8\sqrt{\Phi^*}}-\frac{1}{32}\frac{(\Phi''+D)\,(\Phi''-D)}{\Phi^{*\frac{3}{2}}}+\tfrac{1}{4}\eta L,\quad u=-\tfrac{1}{2}\sqrt{\Phi^*}$$
$$v_+=-\frac{1}{8}\frac{\Phi''}{\sqrt{\Phi^*}},\quad v_-=-\frac{1}{8}\frac{D}{\sqrt{\Phi^*}},\quad w=-\tfrac{1}{4}\eta K'$$

so that Melkich's expression is equivalent to λ^* and likewise the second integral in the formula for g can be transformed into $2h^\dagger$.

A distortion: $\Delta x_b = l x_o y_o^2$

$$l=\frac{x_\gamma}{\sqrt{\Phi_o^*}}\int_{z_B}^{z_b}\frac{x_\gamma}{\sqrt{\Phi^*}}\{\tfrac{1}{4}(\Phi''+D)\,x''_\gamma y_\gamma^2-\Phi^* x''_\gamma y'^2_\gamma$$
$$+\tfrac{1}{8}(\Phi^{(3)}+D')\,x'_\gamma y_\gamma^2+\tfrac{1}{4}(\Phi''+D)\,x_\gamma y'^2_\gamma-\tfrac{1}{2}\Phi' x'_\gamma y'^2_\gamma$$
$$+\Phi^* x'_\gamma y'_\gamma y''_\gamma-\tfrac{1}{2}(\Phi''-D-3\eta K\sqrt{\Phi^*})\,x_\gamma y'^2_\gamma$$
$$-\tfrac{1}{8}[(\eta K/\sqrt{\Phi^*})\,(\Phi''+D)+2\eta(K''-L)\sqrt{\Phi^*}+G]\,x_\gamma y_\gamma^2-\eta K'\sqrt{\Phi^*}\,x_\gamma y_\gamma y'_\gamma\}\,\mathrm{d}z$$
$$-\frac{x_{\gamma b}}{\sqrt{\Phi_o^*}}\int_{z_o}^{z_b}\frac{x_\alpha}{\sqrt{\Phi^*}}\{\tfrac{1}{4}(\Phi''+D)\,x''_y y_\gamma^2-\Phi^* x''_\gamma y'^2_\gamma+\tfrac{1}{8}(\Phi^{(3)}+D')\,x'_\gamma x_\gamma^2+\tfrac{1}{4}(\Phi''+D)\,x'_\gamma y_\gamma y'_\gamma$$
$$-\tfrac{1}{2}\Phi' x'_\gamma y'^2_\gamma+\Phi^* x'_\gamma y'_\gamma y''_\gamma-\tfrac{1}{2}(\Phi''-D-3\eta K\sqrt{\Phi^*})\,x_\gamma y'^2_\gamma$$
$$-\tfrac{1}{8}[(\eta K/\sqrt{\Phi^*})\,(\Phi''+D)+2\eta(K''-L)\sqrt{\Phi^*}+G]\,x_\gamma y_\gamma^2-\eta K'\sqrt{\Phi^*}\,x_\gamma y_\gamma y'_\gamma\}\,\mathrm{d}z.$$

Substituting

$$-\frac{1}{2}\int_B^b\frac{\Phi'}{\sqrt{\Phi^*}}x_\gamma x'_\gamma y'^2_\gamma\,\mathrm{d}z+\tfrac{3}{2}\eta\int_B^b K x_\gamma^2 y'^2_\gamma\,\mathrm{d}z-\int_B^b\sqrt{\Phi^*}\,x_\gamma x''_\gamma y'^2_\gamma\,\mathrm{d}z+\int_B^b\sqrt{\Phi^*}\,x_\gamma x''_\gamma y'^2_\gamma\,\mathrm{d}z$$
$$=\frac{3}{8}\int_B^b\frac{\Phi''-D}{\sqrt{\Phi^*}}x_\gamma^2 y'^2_\gamma\,\mathrm{d}z-\frac{1}{2}\int_B^b\sqrt{\Phi^*}\,x'^2_\gamma y'^2_\gamma\,\mathrm{d}z+\tfrac{1}{2}[\sqrt{\Phi^*}\,x_\gamma x'_\gamma y'^2_\gamma]_B^b,\qquad (C\,13)$$

$$-\eta\int_B^b K' x_\gamma^2 y_\gamma y'_\gamma\,\mathrm{d}z=\tfrac{1}{4}\eta\int_B^b\{(x_\gamma^2)'\,y_\gamma^2-x_\gamma^2(y_\gamma^2)'\}\,\mathrm{d}z+\tfrac{1}{4}\eta\int_B^b K'' x_\gamma^2\,\mathrm{d}z-\tfrac{1}{4}\eta[K' x_\gamma^2 y_\gamma^2]_B^b,$$

$$-\tfrac{1}{8}\eta\int K\frac{\Phi''+D}{\Phi^*}x_\gamma^2 y_\gamma^2\,\mathrm{d}z=-\frac{1}{8}\int\frac{\Phi''+D}{\Phi^*}[\Phi^* x''_\gamma+\tfrac{1}{2}\Phi' x'_\gamma+\tfrac{1}{4}(\Phi''-D)\,x_\gamma]\,x_\gamma y_\gamma^2\,\mathrm{d}z\qquad (C\,14)$$

and

$$\frac{1}{8}\int_B^b\frac{\Phi''+D}{\sqrt{\Phi^*}}x_\gamma x''_\gamma y_\gamma^2\,\mathrm{d}z=\frac{1}{8}\left[\frac{\Phi''+D}{\sqrt{\Phi^*}}x_\gamma x'_\gamma y_\gamma^2\right]_B^b$$
$$-\frac{1}{8}\int\left\{\frac{\Phi'''+D'}{\sqrt{\Phi^*}}x_\gamma x'_\gamma y_\gamma^2-\frac{\Phi''+D}{2\Phi^{*\frac{3}{2}}}\Phi' x_\gamma x'_\gamma y_\gamma^2+\frac{\Phi''+D}{\sqrt{\Phi^*}}x'^2_\gamma y_\gamma^2+2\frac{\Phi''+D}{\sqrt{\Phi^*}}x_\gamma x'_\gamma y_\gamma y'_\gamma\right\}\mathrm{d}z\qquad (C\,15)$$

and collecting up, we find that Melkich's integral from z_B to z_b corresponds to $2e^*$, and likewise the other integral can be transformed into $\lambda^\dagger$.

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